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Statically Indeterminate Structures

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PREFACE

The contents of this book have been selected largely from material developed for courses in the analysis of statically indeterminate structures for senior and graduate students at the University of Michigan. The methods of analysis that are explained and illustrated are based on fundamental principles of structural mechanics that are applicable to the design of most frame structures. In the application of the fundamental principles, particular emphasis is accorded to numerical solutions by various methods of successive approximations, such as moment distribution, iteration, trigonometric series, and the panel method. The analysis of indeterminate structures by means of successive approximations instead of by the more laborious methods that require the solution of many simultaneous equations is undoubtedly the most notable advancement in structural design in the past two decades.

Although the numerical work may be performed by methods of successive approximations, the student must be able to express the relationship of the various physical factors in both geometric and algebraic form. In other words he must thoroughly understand the various methods for calculating displacements in all types of structures and be proficient in their use. This work will naturally precede the study of deformation equations, in which the redundant forces are expressed in terms of displacements. Once the student is familiar with the deformation equations, the principle of superposition, and the general reciprocal theorem, he should have no great difficulty in developing a facility in their application which will enable him to solve most structural problems.

I realize that practically all the fundamental principles that have been employed in this book were developed years ago by engineers to whom the engineering profession is greatly indebted. Among those who should be mentioned particularly are Maxwell, Mohr, Müller-Breslau, Ostenfeld, Castigliano, and Williot. In addition to these earlier engineers, I also wish to acknowledge the influence of more recent work that has been contributed by Maney, Cross, Timoshenko, Southwell, and many others. The publications of these men have been listed in the references at the end of the various chapters, where they can easily be found. It is hoped that the student will use them.

In the arrangement of the subject matter and in checking the numerical solutions, much valuable criticism and assistance have been given by former students, particularly C. W. Pan, G. Hoenke, and P. C. Hu. I wish to express my appreciation to the Portland Cement Association for permission to use the diagrams on pages 321 to 326, to Mr. D. S. Ling for the diagrams on pages 327 to 331, and to many friends and associates for their help and advice.

L. C. Maugh

Ann Arbor, Michigan
February, 1946

CONTENTS

CHAPTER I

CLASSIFICATION AND DESCRIPTION OF STATICALLY INDETERMINATE STRUCTURES

1. Equilibrium Conditions. The classification of structures as statically determinate or indeterminate is usually expressed in mathematical form, although the physical character of the structural framing is the real factor. By structural framing is meant the arrangement and composition of the structural members, the number and characteristics of the supports, and the structural details of the connections. It is assumed that the reader is familiar with the physical laws that force systems must satisfy to maintain a structure in a state of static equilibrium; that is, that coplanar force systems must have no unbalanced components in two arbitrary directions, and no resultant moment about any point. These requirements are commonly expressed in the convenient algebraic form:

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_{xy} = 0 \tag{1}$$

For space frames subjected to a three-dimensional force system, another direction of translation and two additional planes of rotation are possible, and consequently the equilibrium of such structures requires that both equations 1 and 2 be satisfied.

$$\Sigma F_z = 0 \qquad \Sigma M_{xz} = 0 \qquad \Sigma M_{yz} = 0 \tag{2}$$

As the external forces applied to the structure are determined first, they will ordinarily constitute the constants of equations 1 and 2, whereas the reactive forces are the unknowns or variables. When the number of unknown forces are just sufficient to satisfy equations 1 and 2, that is, not more than three for coplanar force systems and not more than six for space frames, the structure is statically determinate. However, only two of the three coplanar forces can be parallel or intersect at a point, and, for space frames, the directions of the reactions must be such that they cannot all be intersected by one straight line, otherwise there is not full restraint. When the number of unknown forces is more than is necessary to satisfy the conditions expressed by equations 1 and 2, the structure is described as statically indeterminate or hyperstatic. The surplus or excess forces are termed

redundant forces, which, although unnecessary for equilibrium, may be desirable for other reasons.

2. Discussion of Redundant Forces. The mathematical requirements stated above form a criterion to be used to establish the degree of indetermination after the number and nature of the external and internal forces have been determined. A careful study of the physical action of each support and connection is essential for a correct determination of the redundant forces that must be considered in the design.

To decide upon the nature of the reactive forces accurate information must be obtained as to the restraint to the motion of the structure that is given by each connection. This restraint can, in general, be classified as resistance to translation, to rotation, or to both. If the amount of this restraint is small, the reactive force produced will also be small and can frequently be neglected. The decision made at this point may actually decide whether the structure is to be considered statically determinate or indeterminate.

A common example in structural framing is the beam shown in Fig. 1*a*, which has total restraint of horizontal and vertical motion at

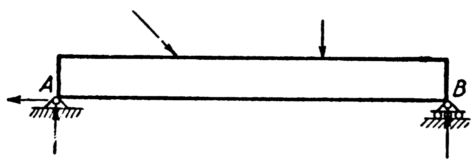


FIG. 1*a*.

point A and of vertical motion at point B. Therefore, as there are only three reactive forces, the beam is statically determinate. If, in addition to these constraints at A and B, points C and D are fastened to rigid supports by the rigid bars shown in Fig. 1*b*, there will be five reactive

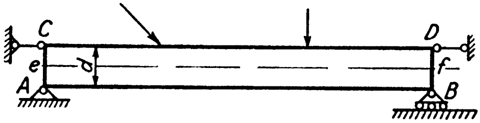


FIG. 1*b*.

forces and, consequently, there will be two redundant forces. The question may well arise as to which two reactive forces should be classified as redundant. In general, the selection is made on the basis of convenience in the mathematical solution.

Another way of representing the forces in Fig. 1*b* is given in Fig. 1*c*. In this form only the axis of the beam is drawn and the reactive forces are replaced by a horizontal force H', a vertical force V_e and moment

M_e at the left end, and a horizontal force H_d, a vertical force V_f, and moment M_f at the right end, where

$$H' = H_A - H_C \qquad M_e = -(H_A + H_C)\frac{d}{2} \qquad M_f = +H_d\frac{d}{2}$$

A positive sign indicates clockwise moment and a negative sign counterclockwise, just as vertical reactions are positive when upward and negative when downward. The end couples or moments should not be regarded as bending moments when the entire member is considered for the same reason that the vertical reactions are not defined as shearing forces until the internal forces are being calculated. The force system shown in Fig. 1*c* constitutes the idealized way of showing the forces in

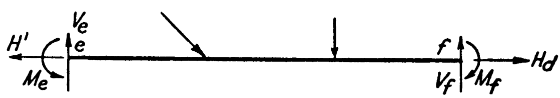


FIG. 1*c*.

Fig. 1*b*. The question will undoubtedly arise in the reader's mind: can such a substitution be made and still sufficient accuracy be obtained? The answer to this question will depend upon the importance of the localized stresses and strains in the vicinity of the concentrated reactive forces as compared to the strain in the entire beam. Certainly the stresses and strains in the material at the ends of the beam for the actual reactions of Fig. 1*b* will be greatly different from those calculated for the idealized force system of Fig. 1*c*. On the other hand, when the internal stresses that exist at a distance from the ends of the beam of more than twice its depth are compared for the actual and idealized conditions, the difference is much less. The magnitude of the error will accordingly depend upon the ratio of the span length to the depth of the beam and upon the portion of the beam over which the strains are computed. If the total strain energy or deformation in a long and flexible span is considered the error is relatively small, but if the beam is short and deep the error may be considerable. The idealized force system is often substituted and used in practical problems without question, but sometimes an investigation of the probable error is necessary, particularly for the stresses near the reactions.

When the entire depth of the beam is fastened to a supporting structure as in Fig. 1*d*, the reactions are distributed over most of the depth of the beam. For this condition the substitution of the idealized force system of Fig. 1*c* is more accurate than the substitution for the forces in Fig. 1*b*. The reason is, of course, that the internal stresses due to

moments and axial forces are calculated on the assumption of a linear distribution over the cross section, and therefore the more nearly the forces are applied in this manner the more accurate are the calculations.

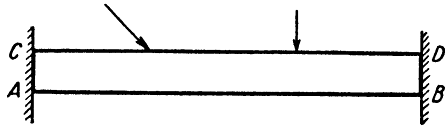


FIG. 1*d*.

When two points on a cross section are restrained against translation, as A and C, Fig. 1*b*, it is usually assumed that the cross section has no rotation. This condition is defined as a fixed end, and the moment M_e for such an end restraint is defined as a **fixed-end moment** (written M_{Fe}). At the end BD, as only one point has horizontal restraint, the end section undergoes some rotation and consequently the moment M_f is a restraining moment but not a fixed-end moment.

3. Internal Redundancy. Structures that are statically determinate with respect to external forces may still be indeterminate with respect to internal forces. An excellent example is the articulated truss in Fig. 2*a*, which, although determinate externally, has more members than are necessary for internal equilibrium. On any section such as *a-a*, Fig. 2*b*, there are four unknown stresses and therefore, for this

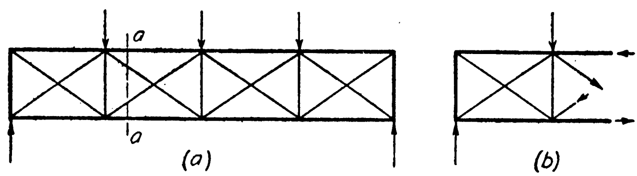


FIG. 2. Statically indeterminate truss.

particular truss, one redundant force. By taking a similar section through each panel it is evident that there is a total of four internal redundant forces in the truss.

Another way in which the number of redundant members in an articulated truss can be determined is by means of the general equilibrium requirement that the total number of bars plus the necessary reactions must be equal to twice the number of joints. This rule can be expressed by the equation

$$n = 2j - 3 \tag{3}$$

in which n = number of members required for stability.
j = number of joints.

The n members must be properly distributed so as to obtain a stable structure.

In space frames each joint requires three bars to maintain equilibrium unless all the forces acting on the joint lie in one plane, in which event only two bars are needed. Therefore, the number of bars plus the number of components of the reactions must be equal to three times the number of joints at which the forces are non-coplanar plus two times the number of joints at which the forces are coplanar or, for the general case,

$$n = 3j_1 + 2j_2 - r \tag{4}$$

where n = number of members for internal stability.
j_1 = joints with non-coplanar forces.
j_2 = joints with coplanar forces.
r = number of components at the supports.

It is possible, of course, to use less than six reactive forces for special types of loading, but most structures should be designed for the general case.

Structures in which the members are subjected to shear and bending moments as well as axial stress cannot, of course, be investigated by equations 3 and 4. Such structures are often many times more redundant internally than externally. The building frame in Fig. 3 is

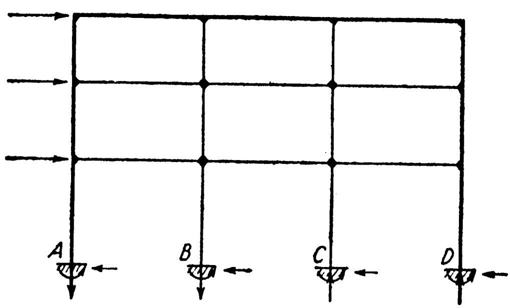


FIG. 3. Statically indeterminate frame.

of this type as it has nine redundant external forces besides nine redundant internal forces in each story above the first. This gives a total of 27 redundant forces for the structure as a whole. In this structure, as in many others, no distinction need be made between internal and external forces as the classification is mainly a matter of location.

4. Strain Conditions. The beam in Fig. 1b is assumed to be held by rigid bars at points A and C which prevent any horizontal displacement of those points. As we shall see in later chapters, it is possible to deter-

mine a force system such that the structure not only is in equilibrium but also will have no horizontal displacements at A and C—in other words, a force system that will satisfy both equilibrium and strain conditions. The presence of H_d makes the problem more complex because of the severe localized strain near D.

In many practical problems, however, the beam will be restrained by connecting pieces that undergo some change in length. For such conditions the movement of C and D will not be as much as in Fig. 1*a* or zero as in Fig. 1*b*, but instead will depend upon the deformation in both beam and connecting bar. The strain in the connections may be an important part of the problem.

As stated previously, the rotation of the cross section can sometimes be used to identify the motion. In Fig. 1*b*, the cross section AC, when restrained at A and C, has no rotation or is fixed, while at the end BD the section rotates through some angle. Now, referring to the idealized force system of Fig. 1*c*, it is possible to identify the position of any point by the linear displacement of the axis of the member and the rotation of the cross section. This use of translation of the axis of the member and rotation of the cross section to determine the actual movement at any point in the member will be sufficiently accurate unless localized effects and shearing deformation are important. It is also important to note that the rotation of the cross section is the same as the rotation of the tangent to the elastic curve if only the curvature due to bending moments is considered, but there is some difference when the shearing detrusion is included.

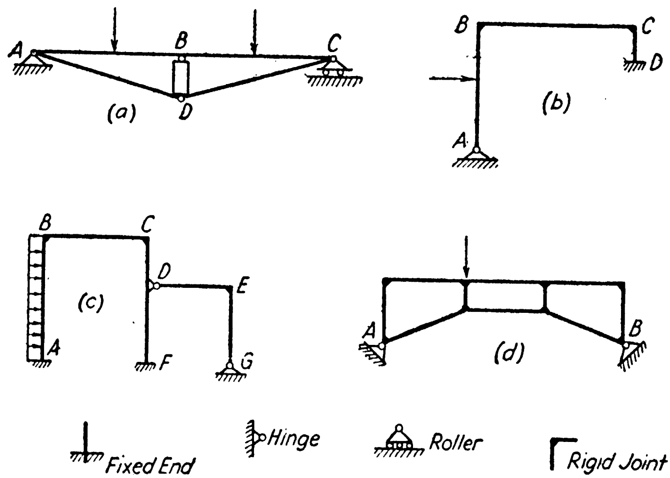
5. Summary. The fact that the analysis of statically indeterminate structures must be based upon a study of the displacements which the various members can undergo, both separately and as a unit, has been emphasized in the preceding articles. At this time warning should be given that the assumption of a certain strain condition presumes experimental evidence to justify it. The designer must study carefully the configuration of actual structures and models to secure accurate data as a basis for his calculations. A study of the results of actual measurements as recorded in technical literature is important in estimating the probable accuracy of analytical solutions. Results that are obtained from mathematical solutions that depend upon unverified assumptions must always be regarded critically. Therein lies an important difference between the equations that are established from equilibrium conditions and those that are obtained from assumed strain conditions, as the former are unquestionable.

For this reason the methods and procedures explained in subsequent chapters for the analysis of various types of statically indeterminate

frame structures give assurance of mathematical accuracy only. The material has been arranged, however, so that the astute observer can incorporate his ideas of the physical action of the structure into the mathematical solution. The design of indeterminate structures requires acumen and sound engineering judgment even more than mathematical ability, but, when carefully conceived, such a structure will often fulfill its function better than a statically determinate one.

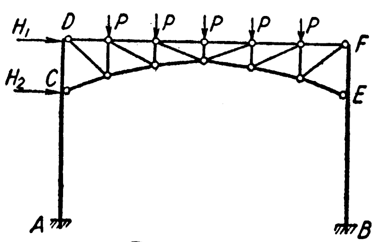
PROBLEMS

1. Determine the number of redundant forces, both external and internal, for the structures shown.



PROBLEM 1.

2. Discuss the degree of redundancy of the transverse bent for the vertical loads P only if the columns are flexible as compared to the truss; also, if the columns are fairly stiff. Do the same for the horizontal loads H.



PROBLEM 2.

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1·2 H. M. Westergaard, "One Hundred Fifty Years' Advance in Structural Analysis," *Trans. Am. Soc. C. E.*, Vol. 94 (1930).

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CHAPTER II

FUNDAMENTAL PRINCIPLES OF STRUCTURAL MECHANICS

6. Hooke's Law: Principle of Superposition. The derivation of most fundamental formulas in mechanics of materials begins with a statement that the unit strain is assumed to be proportional to the unit stress, or displacements proportional to the load. This linear relationship obviously requires an elastic material, and the deformation of such material produces elastic displacements. As this assumption, although seldom exact, is the starting point for the mathematical solution of many problems, it seems desirable to repeat the fact that this stress-strain relation, known as Hooke's law, plays an important role in all subsequent analyses. A study of the stress-strain diagrams for various materials under different loading conditions (that are given in many textbooks on strength of materials) will indicate the accuracy of this assumption.

The **principle of superposition** is useful in structural analysis as it frequently provides simplification of the mathematical work. Primarily this principle states that the stresses and deformations due to any number of forces can be obtained by adding the effects of separate equilibrated force systems and that these forces can be considered in any order. The correct use of this principle requires that two conditions be satisfied: first, that the deformations due to the various loads be computed with the same physical constants, or that Hooke's law holds; and second, that the deformations due to one force system do not affect the deformations caused by another. If either of these conditions is violated, the order in which the loads are considered will affect the final result.

An important problem in which the principle of superposition cannot be directly applied is found in the solution of flexible members that are subjected to both transverse and axial loads. Here, the stresses caused by the axial load are modified by the curvature of the member which must be considered in the calculations. This action means that the displacements caused by one force system will affect those of another. Similarly, as the deflection of a cable of a suspension bridge affects the magnitude of the internal forces, the values of stresses and deformations that are obtained by considering the external forces separately are

therefore not exact. In the following derivations, the principle of superposition is assumed to apply unless stated otherwise.

7. The Reciprocal Theorem. When Hooke's law and the principle of superposition are applicable, another important theorem, the Maxwell-Mohr reciprocal theorem, is readily deduced. This theorem, which was stated briefly by Clerk Maxwell in 1864 and developed in more detail by Otto Mohr in 1874, can be stated as follows:

If a structure is acted upon by two equilibrated force systems P_1, P_2, P_3 ⋯ and F_1, F_2, F_3 ⋯ in which the forces P produce displacements Δ in the direction of the F forces, and the F forces cause displacements y in the direction of the P forces, then the P forces times the corresponding displacements y will be equal to the F forces times the corresponding displacements Δ. In algebraic terms, this theorem is expressed by the equation:

$$P_1y_1 + P_2y_2 + P_3y_3 + \cdots = F_1\Delta_1 + F_2\Delta_2 + F_3\Delta_3 + \cdots \quad (5)$$

That the above reciprocal relation follows from the assumptions previously made can be proved by means of another fundamental principle, the law of conservation of energy. Let us consider two force systems that are applied to the beam in Fig. 4, the first system consisting of the

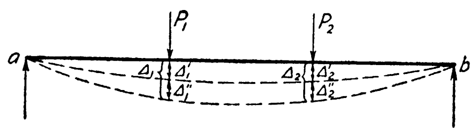


FIG. 4.

load P_1, together with the necessary reactions at a and b, and the second system consisting of the load P_2 and reactions. The load P_1 produces displacements Δ_1' and Δ_2'; the load P_2 produces displacements Δ_1'' and Δ_2''. It will now be proved that

$$P_1\Delta_1'' = P_2\Delta_2'$$

and that the total external work W is equal to

$$\tfrac{1}{2}P_1(\Delta_1' + \Delta_1'') + \tfrac{1}{2}P_2(\Delta_2' + \Delta_2'')$$

or

$$W = \tfrac{1}{2}P_1\Delta_1 + \tfrac{1}{2}P_2\Delta_2$$

Apply load P_1 with displacements Δ_1' and Δ_2'; then the work done equals

$$\tfrac{1}{2}P_1\Delta_1'$$

Now add P_2 with displacements Δ_1'' and Δ_2''; the total work equals

$$\tfrac{1}{2}P_1\Delta_1' + P_1\Delta_1'' + \tfrac{1}{2}P_2\Delta_2''$$

If the load P_1 is now removed, the total work done should be the same, for an elastic material, as though only load P_2 had been placed on the beam, or

$$\tfrac{1}{2}P_1\Delta_1' + P_1\Delta_1'' + \tfrac{1}{2}P_2\Delta_2'' - \tfrac{1}{2}P_1\Delta_1' - P_2\Delta_2' = \tfrac{1}{2}P_2\Delta_2''$$

from which

$$P_1\Delta_1'' = P_2\Delta_2'$$

If the loads P_1 and P_2 are applied simultaneously, the total work done will be identical with that obtained by first applying P_1 and then P_2, or

$$W = \tfrac{1}{2}P_1\Delta_1' + P_1\Delta_1'' + \tfrac{1}{2}P_2\Delta_2''$$

but

$$P_1\Delta_1'' = \tfrac{1}{2}P_1\Delta_1'' + \tfrac{1}{2}P_1\Delta_1'' = \tfrac{1}{2}P_1\Delta_1'' + \tfrac{1}{2}P_2\Delta_2'$$

since

$$P_1\Delta_1'' = P_2\Delta_2'$$

hence,

$$W = \tfrac{1}{2}P_1(\Delta_1' + \Delta_1'') + \tfrac{1}{2}P_2(\Delta_2' + \Delta_2'') = \tfrac{1}{2}P_1\Delta_1 + \tfrac{1}{2}P_2\Delta_2 \qquad (6)$$

The same relations can now be extended, by a similar procedure, to force systems consisting of several forces. For example, the force system P_1, P_2 in Fig. 5 produces the displacements Δ_1, Δ_2, Δ_m, θ_a while

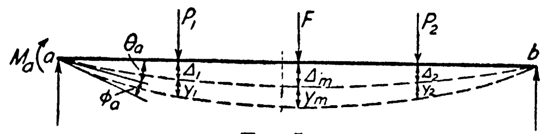


FIG. 5.

the second system F, M_a, causes the displacements y_1, y_2, y_m, ϕ_a. The work done during the application of P_1 and P_2 has just been shown to be

$$W_1 = \tfrac{1}{2}P_1\Delta_1 + \tfrac{1}{2}P_2\Delta_2$$

When the second system F, M_a, is applied, the work done is

$$W_2 = P_1y_1 + P_2y_2 + \tfrac{1}{2}Fy_m + \tfrac{1}{2}M_a\phi_a$$

If the first system is removed and the material is assumed to be perfectly elastic, the work performed is

$$W_3 = -M_a\theta_a - F\Delta_m - \tfrac{1}{2}P_1\Delta_1 - \tfrac{1}{2}P_2\Delta_2$$

Now, by the principle of superposition, the energy remaining in the beam after the removal of P_1, P_2, should be the same as though only F, M_a had been applied, and, therefore,

$$W_1 + W_2 + W_3 = \tfrac{1}{2}Fy_m + \tfrac{1}{2}M_a\phi_a$$

Substituting the values of W_1, W_2, W_3 in the above expression gives

$$P_1y_1 + P_2y_2 = F\Delta_m + M_a\theta_a$$

which is the same relation as expressed by equation 5.

The reciprocal theorem is an important tool in the mechanical analysis of complicated structures by means of small-scale models. The following application to the solution of a continuous beam will illustrate its use.

Example 1. An influence diagram for the reaction R_a (value of R_a for any value of x) of the continuous beam $ABCD$ (Fig. 6a) will be obtained by means of a small-scale model.

Solution. Construct a small-scale model as shown in Fig. 6b in which the span lengths and moments of inertia are proportional to the values

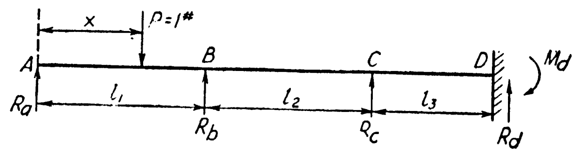


FIG. 6a.

in the actual beam. The model should be mounted on ball bearings and supported as shown. If a transverse force F_1 is applied, the axis of the model will take some curve as indicated by the dotted line. The actual values of these displacements can be measured with a microscope or micrometer screw. Now let us regard the unit load P and the accompanying reactions (Fig. 6a) as one force system and the loads F_1, F_2, F_3

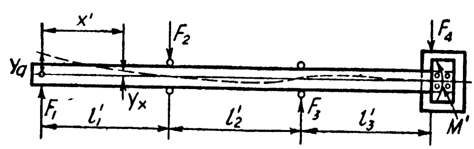


FIG. 6b.

that are actually applied to the model (Fig. 6b) as a second force system. Then, by the reciprocal theorem, equation 5, we obtain

$$-(1\text{ lb})(y_x) + R_a y_a + R_b(0) + R_c(0) + R_d(0) + M_d(0)$$
$$= F_1(0) + F_2(0) + F_3(0) + F_4(0) + M'(0)$$

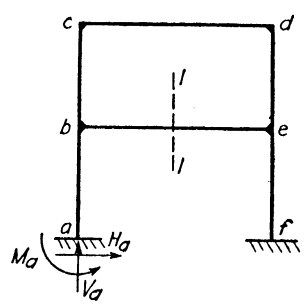
or

$$R_a = 1\text{ lb}\,\frac{y_x}{y_a} \tag{7}$$

The ordinates to the influence diagram for R_a can therefore be obtained by measuring the displacements y to the elastic curve produced by the F forces. In general, any influence diagram for a redundant external or internal force can be obtained from some elastic curve, but *not* every elastic curve is an influence diagram. The best test is to apply the reciprocal theorem. The fact that no forces need be measured makes the above procedure a valuable tool for the analysis of extremely complicated structures by means of small-scale flexible models.

PROBLEMS

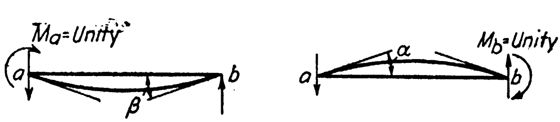
3. (a) Explain fully how influence diagrams for H_a, V_a, M_a for the frame *abcdef* can be obtained by means of a model. Can the values obtained from the model be applied directly to the prototype?



PROBLEM 3.

(b) How would you obtain an influence diagram for the bending moment at section 1–1 by means of a model? If section 1–1 is at mid-span, sketch the shape of the influence diagram that you think you would obtain.

4. By means of the reciprocal theorem, show that the angular rotation β at b for a unit moment at a is the same as the rotation α at a for a unit moment at b. Is this statement still valid when the cross section of the beam varies?



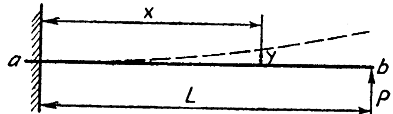
PROBLEM 4.

5. If the value of y (Fig. a) is given by the equation

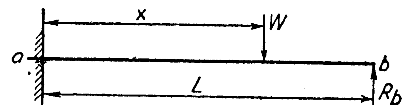
$$y = \frac{Px^2}{6EI}(3L - x)$$

show by the reciprocal theorem that the reaction R_b (Fig. b) is expressed by the equation

$$R_b = \frac{Wx^2}{2L^3}(3L - x)$$



(a)



(b)

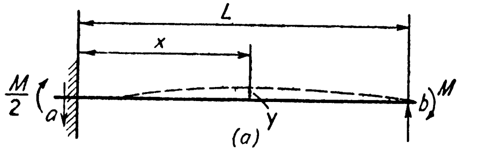
PROBLEM 5.

6. If the value of y (Fig. a) is given by the equation

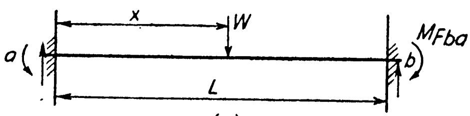
$$y = \frac{Mx^2}{4EI}\left(1 - \frac{x}{L}\right)$$

show by the reciprocal theorem that the fixed-end moment M_{Fba} (Fig. b) is expressed by the equation

$$M_{Fba} = \frac{Wx^2}{L}\left(1 - \frac{x}{L}\right)$$



(a)



(b)

PROBLEM 6.

8. Internal Deformation and Strain Energy. A structural member that must resist the action of applied loads is subjected on any normal section to a resultant force system that can be resolved into a bending moment M, a transverse or shearing force V, an axial or normal force N,

and a twisting moment or torque T. As formulas for the unit stresses and strains due to these internal forces are treated in any standard textbook on the strength of materials, they will not be repeated here. However, for future use it will be convenient to have a summary of the strain energy that such force systems cause in the material.

In a straight prismatic member that is subjected to flexure (Fig. 7*a*), each section tends to rotate about a principal axis of inertia when the bending moment M is applied in a plane that contains a principal axis.

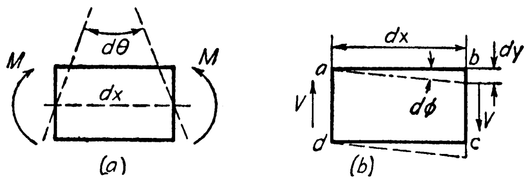


FIG. 7*a* and *b*. Rotation due to bending moments and translation due to shear in an element of a beam.

The magnitude of the relative rotation of two normal sections, a distance dx apart, is $d\theta = \dfrac{M\,dx}{EI}$, and the strain energy dU stored in the material will be equal to

$$dU = \frac{M}{2}\,d\theta = \frac{M^2\,dx}{2EI} \tag{8a}$$

The strain energy caused by the shear V is somewhat complicated by the variable distribution of shearing stress over the cross section. If the shearing stress s_s were uniformly distributed, the relative transverse displacement dy (Fig. 7*b*) would equal

$$dy = \frac{s_s}{G}\,dx = \frac{V}{GA}\,dx$$

and the work done is

$$\frac{V}{2}\,dy = \frac{V^2}{2GA}\,dx$$

However, as the shearing stress usually varies over the section, the correct expression for strain energy is

$$dU = k\,\frac{V^2\,dx}{2GA} \tag{8b}$$

in which G is the modulus of rigidity; A, the cross-sectional area; and k, a constant that takes into account the variation of the unit shearing stress. The value of k is difficult to evaluate except for rectangular

cross sections, for which it equals 1.2. When the shearing stress is not uniform, the angle of detrusion $d\phi$ will vary across the section. For an I-section, A should be taken equal to the web area only and k equal to unity.

An axial force N (Fig. 7c) will produce an extension or contraction of the member equal to $\frac{N}{AE}$ per unit of length, and a total deformation for the entire member of

$$\Delta = \frac{Nl}{AE}$$

if the value of N is constant. The elastic strain energy will then equal

$$U = \tfrac{1}{2}N\Delta = \frac{N^2l}{2AE} \tag{8c}$$

If N is not constant throughout, the strain energy can be obtained by summation or integration.

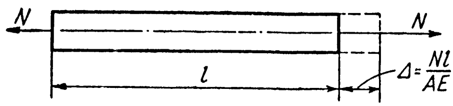


FIG. 7c. Change in length due to axial stress.

A torque T produces a rotation or twist about a polar axis through the centroid of the section. As this twist causes a warping of the section except for circular cross sections, the value of this rotation, $d\phi$, is not easy to obtain mathematically for non-circular members. For a circular section

$$d\phi = \frac{T\,dx}{GJ}$$

where J is the polar moment of inertia, and the strain energy is

$$dU = \frac{T\,d\phi}{2} = \frac{T^2\,dx}{2GJ} \tag{8d}$$

For rectangular sections, the value of J is not the polar moment of inertia, but equation 8d can be used if J is taken equal to

$$J = \frac{b^3d^3}{3.58(b^2 + d^2)}$$

The total strain energy U in any structural member is obtained by combining the energy due to the various components, and the total

internal energy in a structure is obtained by the summation of the strain energy in all members.

9. Deflection of Beams by Equating External and Internal Work. The work done by external forces acting through elastic displacements and the internal work performed by the internal stresses acting through the deformation of the fibers, which may be considered strain energy stored in the material, are quantities that are frequently used in the solution of structural problems. As the nature and magnitude of this strain energy for elastic conditions have already been discussed, we are now ready to develop some of the methods by which it can be applied.

The law of the conservation of energy has been mentioned as an important principle in engineering science, and, in calculating some displacements, a numerical solution can be made by equating the actual external work to the corresponding internal strain energy. In general, however, if the external work due to two forces such as P_1 and P_2, Fig. 4, is set equal to the corresponding internal energy caused by the bending moments only, the equation

$$\tfrac{1}{2}P_1\Delta_1 + \tfrac{1}{2}P_2\Delta_2 = \int_a^b \frac{M^2\,dx}{2EI} = U \tag{9}$$

is obtained. The right side of this equation can be evaluated, but, as there are two unknowns Δ_1 and Δ_2 on the left side, no solution is possible. If only *one* displacement is involved, a numerical solution can be made. This desirable condition can always be produced by the simple expedient of placing an *auxiliary* (frequently called a virtual) force system on the structure before the actual forces are applied. Then, when the actual forces are applied, the **external work done by the auxiliary force acting through the actual displacement is equated to the corresponding internal energy due to the internal auxiliary forces times the actual internal displacements.** The auxiliary force system must necessarily consist of a force that is applied at the point and in the *direction* of the displacement that is desired, together with the reactions that are necessary to form an equilibrated force system.

For example, if the deflection at the center of the beam in Fig. 8*a* is desired, the auxiliary force system that is shown in Fig. 8*b* would be applied first and would therefore act through the displacements caused by the actual loads. By the law of the conservation of energy, the external work that is done by the auxiliary load P' acting through the real displacement Δ must equal the internal work done by the auxiliary moments M' acting through the real rotations $d\theta$ caused by M. By

taking advantage of symmetry, this relation is expressed by the equation

$$P'\Delta = 2\int_0^{l/2} M'\,d\theta = 2\int_0^{l/2} M'\frac{M\,dx}{EI} \tag{10a}$$

If P' is assumed equal to unity and M' equal to m, the moment when P' is equal to unity, the above theorem takes the form

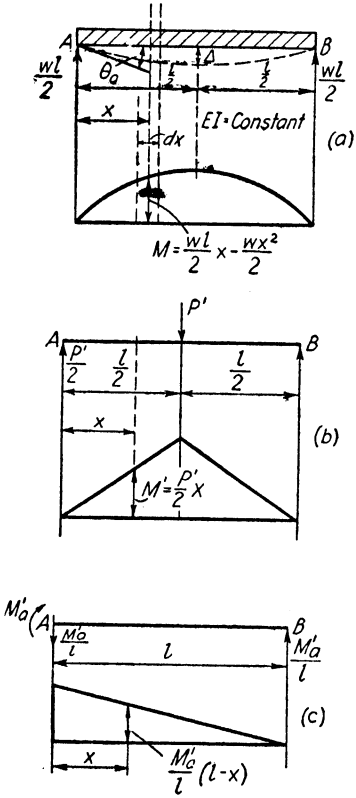


FIG. 8.

$$1\text{ lb}\cdot\Delta = \int\frac{Mm\,dx}{EI} \tag{10b}$$

It is important to note that equation 10*b* involves the external and internal work done by *assumed* forces acting through *real* displacements.

If the values of M' and M, in terms of x, Figs. 8*a* and *b*, are substituted in equation 10*a*, the force P' can be canceled out and the value of Δ becomes

$$P'\Delta = \frac{P'}{2EI}\int_0^{l/2}(x)(wlx - wx^2)\,dx$$

or

$$\Delta = \frac{5}{384}\frac{wl^4}{EI}$$

Equation 10*b*, for P' equal to unity, gives the same result.

If the rotation θ_a of the end tangent at A is desired, the auxiliary force system shown in Fig. 8*c* would be used. The external and internal work resulting from these assumed forces acting through the real displacements is found from the expression:

$$M'_a\theta_a = \int_0^l M'_x\,d\theta = \int_0^l M'_x\frac{M\,dx}{EI} \tag{10c}$$

or

$$M'_a\theta_a = \frac{M'_a}{EI}\int_0^l\left(\frac{l-x}{l}\right)\left(\frac{wlx}{2} - \frac{wx^2}{2}\right)dx$$

from which

$$\theta_a = \frac{wl^3}{24EI}$$

Although the above method for calculating the deflection of beams can be used for any type of loading, the numerical work is laborious except for simple problems, and, therefore, the conjugate-beam method that will be described later is recommended for more complex problems. The effect of shearing strain upon the displacement can be included by adding the corresponding internal work done by the shearing forces to the right side of equation 10*b*:

$$1\text{ lb }\Delta = \int_0^L \frac{Mm\,dx}{EI} + \int_0^L k\frac{Vv\,dx}{GA} \tag{11}$$

in which v is the shear due to the unit auxiliary force.

10. Deflection of Trusses by Equating External and Internal Work. For the calculation of deflections in trusses in which the internal strain energy is primarily due to axial stress in the various members, equating of external and internal work provides a direct and convenient solution. As the only difference between this calculation and the preceding calculation of the deflection of beams is in the expression for the internal energy, the procedure will be illustrated by a numerical example.

Example 2. The movement Δ of point h, Fig. 9*a*, will be calculated for the loads, internal stresses, and areas shown on the truss. An

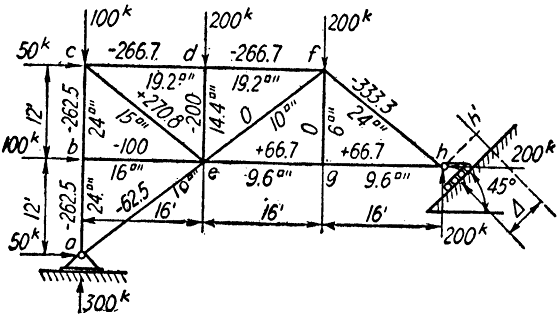


FIG. 9*a*. Stresses S due to actual loads.

auxiliary force P' equal to 1 kip and the necessary reactions (Fig. 9*b*) are assumed to be acting before the actual loads are applied. As for the preceding beam, we can now write:

(Auxiliary external force P') (Actual external displacement Δ) = (Auxiliary internal forces due to P') (Actual internal deformations δL)

or, if P' is taken equal to unity,

$$1\text{ kip}\cdot\Delta = \Sigma u(\delta L) = \Sigma u \cdot \frac{SL}{AE} \tag{12}$$

where u = stress in any member due to $P' = 1$ kip.

$\frac{SL}{AE} = \delta L$ = change in length of the same member due to actual loads.

The numerical values of these terms are summarized in Table 1. Note that the value of E is taken outside the summation sign as it is a constant.

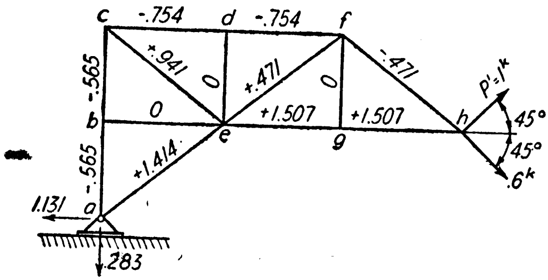


FIG. 9b. Stresses u due to auxiliary load.

TABLE 1

MEMBER	$\frac{L}{A}$	S (kips)	$\frac{SL}{A}$	u (kips)	$u\frac{SL}{A}$
ab	6.0	−262.5	−1575	−0.565	890
ae	24.0	− 62.5	−1500	1.414	−2,120
bc	6.0	−262.5	−1575	−0.565	890
be	12.0	−100.0	−1200	0	0
cd	10.0	−266.7	−2667	−0.754	2,010
ce	16.0	270.8	4340	0.941	4,080
de	10.0	−200.0	−2000	0	0
df	10.0	−266.7	−2667	−0.754	2,010
ef	24.0	0	0	0.471	0
eg	20.0	66.7	1334	1.507	2,010
fg	24.0	0	0	0	0
fh	10.0	−333.3	−3333	−0.471	1,570
gh	20.0	66.7	1334	1.507	2,010

$$\Sigma u \frac{SL}{A} = 13{,}350$$

$$1 \text{ kip}\cdot\Delta = \frac{1}{E}\Sigma u \frac{SL}{A} = \frac{13{,}350 \text{ kips}^2/\text{in.}}{29{,}000 \text{ kips/in.}^2}$$

or

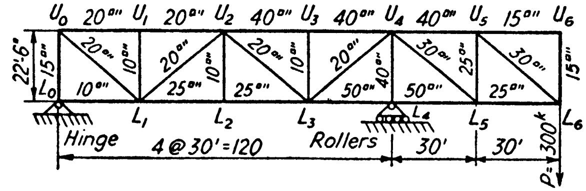
$$\Delta = 0.46 \text{ in.}$$

Again it should be noted that the above solution involves the external and internal work done by an auxiliary or virtual load P' acting through

the corresponding *actual* displacements of the structure. This method of solution is extremely useful when the displacements of only one or two points are required, but the graphical solution by means of the Williot diagram, presented later, is more convenient for obtaining the entire elastic curve of the structure.

PROBLEMS

7. Compute the vertical displacement of joint L_6 for the load $P = 300$ kips and for the areas of the members that are recorded on the truss. (Use $E =$ 30,000,000 lb per in.^{2}) *Ans.* 4.6 in.

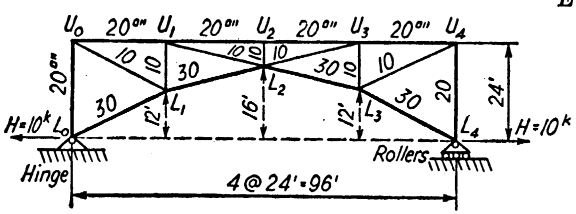


PROBLEM 7.

8. Compute the vertical displacement of joint L_2 for the truss in Problem 7. *Ans.* 0.9 in. (upward).

9. Compute the horizontal displacement of joint U_6 for the truss in Problem 7. *Ans.* 0.53 in.

10. Determine the horizontal displacement of L_4 and the vertical displacement of U_2 in terms of E. The areas of the members are given in the diagram. All values are expressed in kips. *Ans.* $\dfrac{4094}{E}$ and $\dfrac{4706}{E}$.



PROBLEM 10.

11. From the answers obtained in Problem 10, determine the value of the horizontal reaction H for a vertical load of 40 kips applied at U_2, if the structure has a hinge at L_4 similar to the one at L_0. *Ans.* $H = 46$ kips.

11. Castigliano's Theorem. A general method for computing displacements is given by an important relation between forces and strain energy that is known as Castigliano's theorem. To explain the meaning of this theorem, let us first consider the loads and displacements of

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the beam in Fig. 10. If the loads P_2 and P_3 are applied to the beam first, the elastic curve will take some position as shown by the ordinates y. When the load P_1 is placed on the beam, the elastic curve will undergo additional displacements as shown by the ordinates z. If the load P_1 is now increased by some increment dP_1, the displacements z_1, z_2, z_3 will be changed by the amounts dz_1, dz_2, dz_3.

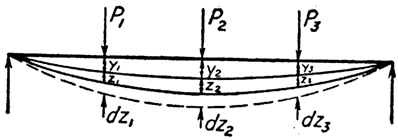


FIG. 10.

The change in the strain energy dU due to the addition of the increment dP_1 is

$$dU = \left(P_1 + \frac{dP_1}{2}\right) dz_1 + P_2\, dz_2 + P_3\, dz_3$$

If Hooke's law holds, then

$$dz_1 = \frac{z_1}{P_1}\, dP_1 \qquad dz_2 = \frac{z_2}{P_1}\, dP_1 \qquad dz_3 = \frac{z_3}{P_1}\, dP_1$$

or

$$dU = \tfrac{1}{2} dP_1\, dz_1 + \frac{1}{P_1}(P_1 z_1 + P_2 z_2 + P_3 z_3)\, dP_1$$

But, by the reciprocal theorem,

$$P_2 z_2 + P_3 z_3 = P_1 y_1$$

or, neglecting the term $\tfrac{1}{2} dP_1\, dz_1$,

$$dU = \left(\frac{P_1 z_1 + P_1 y_1}{P_1}\right) dP_1 = \Delta_1\, dP_1$$

which is usually written

$$\frac{dU}{dP_1} = \Delta_1 \qquad (13a)$$

where Δ_1 is the total displacement in the direction of P_1.

Castigliano's theorem, as represented by equation 13a, can be stated as follows:

When a structure is acted upon by an equilibrated force system which produces a total internal strain energy U, the derivative of U with respect to any force gives the displacement in the direction of that force.

A more direct interpretation of equation 13*a* is obtained from Fig. 11, in which the displacement Δ_1, due to all forces on the structure, varies linearly as only the force P_1 is increased. The intercept on the Y axis gives the value of Δ_1 when P_1 is equal to zero. If the force P_1 is increased from any value by an amount dP_1, the shaded area, which

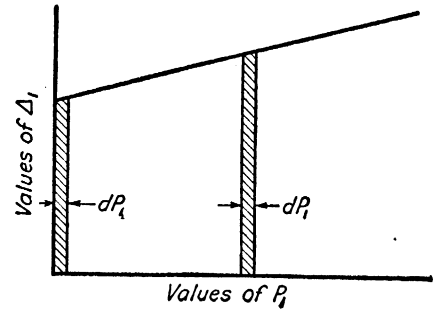


FIG. 11.

represents the external work done, must be equal to the change in strain energy dU in the structure. That is,

$$\Delta_1 \cdot dP_1 = dU$$

or

$$\Delta_1 = \frac{dU}{dP_1} \tag{13a}$$

Obviously, as this relation holds for any value of P_1, it can be used when P_1 is equal to zero; that is, the increment of load can start from zero. Consequently, if the displacement is desired at a point where no load is applied, a force P_1 must be assumed in the direction of the displacement and its **magnitude reduced to zero after the algebraic expression for the strain energy has been differentiated.** This operation implies that, although P_1 may be set equal to zero, $\dfrac{dU}{dP_1}$ is not.

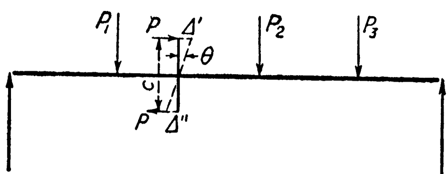


FIG. 12.

If a couple M equal to Pc, Fig. 12, is applied, the work done by any increase dP in both forces P is

$$dU = (\Delta' + \Delta'')\,dP = \frac{(\Delta' + \Delta'')}{c} \cdot c \cdot dP$$

But $\frac{\Delta' + \Delta''}{c}$ is equal to the rotation θ, and $c \cdot dP$ is the increase dM of the applied couple. Therefore any rotation θ can be calculated from the relation

$$\theta = \frac{dU}{dM} \tag{13b}$$

The numerical arrangement obtained from equations 13*a* and *b* is, in its final form, identical with the equations previously established by equating external and internal work. Castigliano's theorem will frequently provide the most convenient method of analysis when the solution is expressed in general algebraic terms; but, for most problems, the amount of work involved in the two methods is about the same.

Example 3. The application of Castigliano's theorem to the determination of displacements in structures will be exemplified by the calculation of the vertical displacement Δ_B of point B and the horizontal movement Δ_C of point C for the semicircular arch in Fig. 13*a*. The internal forces acting upon any right section of the arch at an angle θ with the horizontal are shown in Fig. 13*b*. The magnitudes of these forces are

$$N = \frac{P}{2}\cos\theta \qquad V = \frac{P}{2}\sin\theta \qquad M = \frac{Pr}{2}(1 - \cos\theta)$$

The expression for the total strain energy U in the structure will therefore be (note symmetry)

$$U = \int_A^C \frac{M^2\,ds}{2EI} + \int_A^C \frac{V^2\,ds}{2AG} + \int_A^C \frac{N^2\,ds}{2AE}$$

$$U = 2\int_0^{\pi/2} \frac{\left[\frac{Pr}{2}(1 - \cos\theta)\right]^2 r\,d\theta}{2EI} + 2\int_0^{\pi/2} \frac{\left(\frac{P}{2}\sin\theta\right)^2 r\,d\theta}{2AG} + 2\int_0^{\pi/2} \frac{\left(\frac{P}{2}\cos\theta\right)^2 r\,d\theta}{2AE}$$

since $\frac{dU}{dP} = \Delta_B$,

$$\Delta_B = \frac{1}{EI}\int_0^{\pi/2} \frac{P[r(1 - \cos\theta)]^2 r\,d\theta}{2} + \frac{1}{AG}\int_0^{\pi/2} \frac{P\sin^2\theta\, r\,d\theta}{2} + \frac{1}{AE}\int_0^{\pi/2} \frac{P\cos^2\theta\, r\,d\theta}{2}$$

from which

$$\Delta_B = 0.178\frac{Pr^3}{EI} + 0.393\frac{Pr}{AG} + 0.393\frac{Pr}{AE}$$

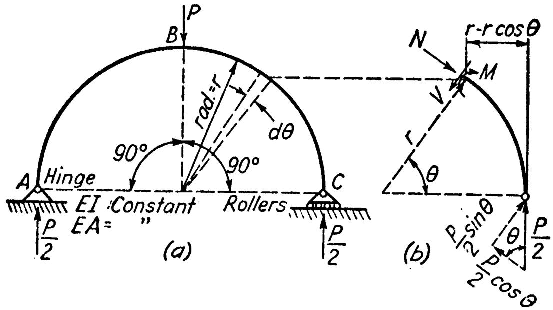


FIG. 13a and b.

Let us assume that the arch is composed of an I-section with the following properties:

$$I = 214 \text{ in.}^4$$

$$\text{Total area } A = 8.23 \text{ in.}^2$$

$$\text{Area of web} = 2.7 \text{ in.}^2$$

$$G = 0.4E \qquad r = 10 \text{ ft} = 120 \text{ in.}$$

If the web area is considered as taking all the shear, then the above value of Δ_B becomes:

$$\Delta_B = \frac{1437P}{E} + \frac{44}{E}P + \frac{6}{E}P = \frac{1487P}{E}$$

A comparison of the relative magnitudes of the three terms shows that the displacement due to the bending moment is about 97 per cent of the total. Consequently, for many arch structures and for many frames, the effect of the shear and direct stress may be neglected.

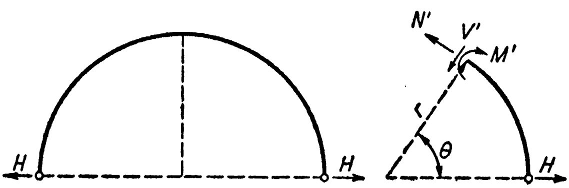


FIG. 13c.

To compute the horizontal displacement Δ_C of point C, the auxiliary force system shown in Fig. 13c will be added to the actual force system

of Fig. 13a. The internal forces at any section due to the load H are

$$N' = H \sin \theta \qquad V' = H \cos \theta \qquad M' = Hr \sin \theta$$

When the force systems of Figs. 13a and 13c are both applied, the total strain energy U will be:

$$U = 2\int_0^{\pi/2} \frac{\left\{\dfrac{Pr(1 - \cos \theta)}{2} + Hr \sin \theta\right\}^2 r\, d\theta}{2EI} +$$

$$2\int_0^{\pi/2} \frac{\left\{\dfrac{P}{2} \sin \theta + H \cos \theta\right\}^2 r\, d\theta}{2AG} + 2\int_0^{\pi/2} \frac{\left\{\dfrac{P}{2} \cos \theta - H \sin \theta\right\}^2 r\, d\theta}{2AE}$$

$$\frac{\partial U}{\partial H} = \Delta_C = \frac{1}{EI}\int_0^{\pi/2} 2\left[\frac{Pr(1 - \cos \theta)}{2} + Hr \sin \theta\right][r \sin \theta]\, r\, d\theta$$
$$+ \frac{1}{AG}\int_0^{\pi/2} 2\left[\frac{P}{2} \sin \theta + H \cos \theta\right] \cos \theta\, r\, d\theta$$
$$+ \frac{1}{AE}\int_0^{\pi/2} 2\left[\frac{P}{2} \cos \theta - H \sin \theta\right][- \sin \theta]\, r\, d\theta$$

To obtain the correct value of Δ_C, the auxiliary force H must be reduced to zero, giving

$$\Delta_C = \frac{1}{EI}\int_0^{\pi/2} Pr^3 (1 - \cos \theta) \sin \theta\, d\theta + \frac{1}{AG}\int_0^{\pi/2} Pr \sin \theta \cos \theta\, d\theta$$
$$- \frac{1}{AE}\int_0^{\pi/2} Pr \sin \theta \cos \theta\, d\theta$$

from which

$$\Delta_C = \frac{Pr^3}{2EI} + \frac{Pr}{2AG} - \frac{Pr}{2AE}$$

Although the practical application of equation 13 is often laborious, nevertheless it is a useful tool in the solution of many problems. The example just solved shows that the numerical work is greatly reduced if the differentiation is performed before the integration. The principal advantage of the method is the ease with which the energy for all internal forces, i.e., moments, shear, axial stress, and torque, can be incorporated into the equations. The disadvantage lies mainly in the laborious task of solving the equations that are often involved.

12. Theorem of Minimum Energy. In the solution of redundant structures, a common method of analysis is to transform certain internal forces into external forces by cutting sections through the struc-

tural members. The strain energy can then be expressed in terms of these unknown internal forces in the same manner as for external forces. As there is no relative motion between the two parts made by the cutting section, then

$$\frac{dU}{dP} = \Delta = 0 \tag{14}$$

which proves that the stresses in a redundant structure will take such values as to make the strain energy a minimum and still provide

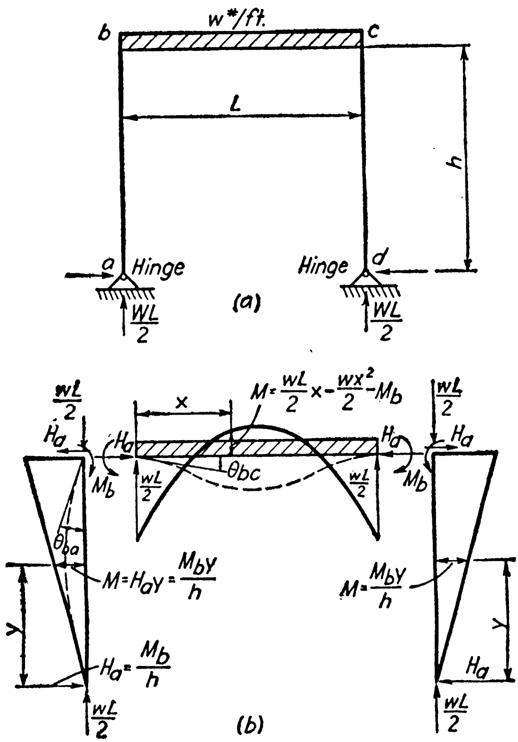


FIG. 14.

equilibrium. In other words, **the configuration of the structure is such that the amount of internal work performed will be the least possible.** The determination of redundant forces by the above method will be illustrated by the analysis of the frame shown in Fig. 14*a*.

Example 4. As the frame is once indeterminate, the internal forces must contain one unknown which is taken as M_b, the bending moment at the corner b. The bending moment in each member is recorded in terms of M_b and w in Fig. 14*b*.

The total strain energy in the frame due to flexure only is

$$U = 2\int_0^h \frac{\left(M_b \frac{y}{h}\right)^2 dy}{2EI} + \int_0^L \frac{\left(\frac{wL}{2}x - \frac{wx^2}{2} - M_b\right)^2 dx}{2EI}$$

If the joint at b is assumed to be rigid, that is if the rotation of the end tangent θ_{bc} of bc is the same as the end rotation θ_{ba} of ba, no relative rotation can occur between the two tangents, and consequently

$$\frac{dU}{dM_b} = \theta_{ba} - \theta_{bc} = 0$$

Therefore

$$\frac{dU}{dM_b} = \frac{2}{EI}\int_0^h \left(\frac{M_b y}{h}\right)\frac{y}{h}dy + \frac{1}{EI}\int_0^L \left(\frac{wL}{2}x - \frac{wx^2}{2} - M_b\right)(-dx) = 0$$

giving

$$\frac{1}{EI}\left(\frac{2M_bh}{3} - \frac{wL^3}{12} + M_bL\right) = 0$$

from which

$$M_b = \frac{wL^2}{12\left(1 + \frac{2h}{3L}\right)}$$

The positive sign obtained for M_b indicates that the direction of the assumed moment is correct.

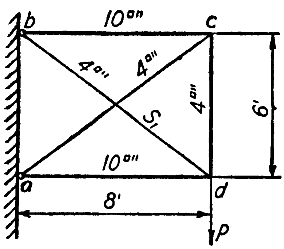


FIG. 15.

Example 5. The redundant stresses in trusses, such as in Fig. 15, can also be solved by equation 14. Thus, if S_1 represents the stress in the diagonal bd, the stresses in the other members are

$$cd = P - 0.6S_1 \qquad ad = -0.8S_1$$

$$ac = -1.67P + S_1 \qquad bc = 1.33P - 0.8S_1$$

The total strain energy $U = \Sigma \dfrac{S^2L}{2AE}$, or

$$U = \frac{12}{2E}\left[\frac{S_1^2(10)}{4} + \frac{(P - 0.6S_1)^2(6)}{4} + \frac{(-0.8S_1)^2(8)}{10} + \frac{(-1.67P + S_1)^2(10)}{4} + \frac{(1.33P - 0.8S_1)^2(8)}{10}\right]$$

$$\frac{dU}{dS_1} = \frac{12}{E}\left[2.5S_1 + \frac{(P - 0.6S_1)(-0.6)(6)}{4} + \frac{(-0.8S_1)(-0.8)(8)}{10} + \frac{(-1.67P + S_1)(10)}{4} + \frac{(1.33P - 0.8S_1)(-0.8)(8)}{10}\right] = 0$$

from which

$$S_1 = 0.902P \quad \text{(tension as assumed)}$$

If the truss contains many members, a tabular arrangement of the calculation should be used. Note that each term contains the expression

$$\frac{S\dfrac{\partial S}{\partial S_1}L}{AE} = \frac{SuL}{AE}$$

where u is the stress in the member for S_1 equal to unity.

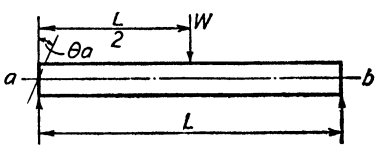
PROBLEMS

12. (*a*) Determine the rotation of the end section due to flexure only. I = constant.

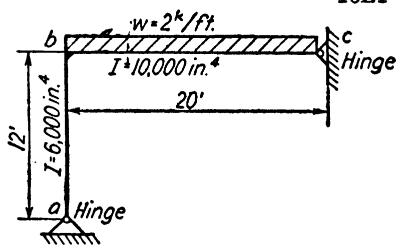
Ans. $\theta_a = \dfrac{WL^2}{16EI}$.

(*b*) Determine the rotation of the end section due to shear only.

Ans. $\theta_a = 0$.



PROBLEM 12.



PROBLEM 13.

13. (*a*) Determine the horizontal reaction at a by means of Castigliano's theorem. Consider the effect of flexure only. *Ans.* 4.17 kips.

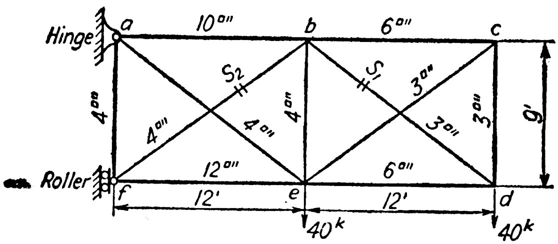
(*b*) Draw the bending-moment diagram and the elastic curve for the frame.

14. Determine the reactions at a for the frame in Problem 13, if the structure is fixed at a and hinged at c.

Ans. $H_a = 7.14$ kips; $M_a = 28.6$ ft-kips; $V_a = 22.86$ kips.

15. Calculate the redundant stresses S_1 and S_2 if all members of the truss are acting. Arrange the calculations in tabular form.

Ans. $S_1 = +37.2$ kips; $S_2 = -57.6$ kips.



PROBLEM 15.

13. Theorems of Area Moments (Relative Displacements in Beams). In the discussion of strain energy due to flexure, the statement was made, and illustrated in Fig. 7a, that two normal sections of a straight beam that are a distance dx apart undergo a relative rotation

$$d\theta = \frac{M\,dx}{EI}$$

when subjected to a bending moment M. The result of such a deformation in one element $abcd$ of a beam is shown in Fig. 16, where the sections ab and cd have a relative rotation $d\theta_1$.

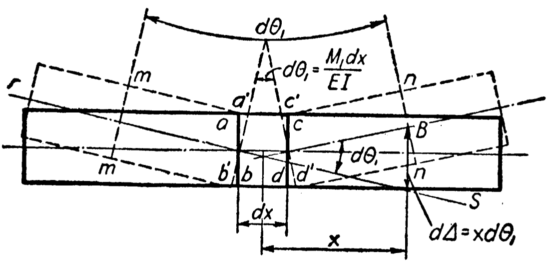


FIG. 16.

From the relative position of the portions of the beam on each side of the element as indicated by the dotted lines, it can be seen that any section m to the left of the element will be rotated through the same angle $d\theta_1$ with respect to any section n on the right side. This rotation does not define the position of either section m or n, however, as we do not know how much sections ab or cd have rotated or translated from

their original positions. It can also be seen that a point B in section n-n will be at a distance of $d\Delta$ equal to $x\,d\theta_1$ from the rotated axis rs on the left side. It is assumed that $d\Delta$ can be measured perpendicular to the original axis of the beam, as the rotations are extremely small. When the deformation of two elements is considered, the relative positions of the various segments are indicated in Fig. 17. Although only

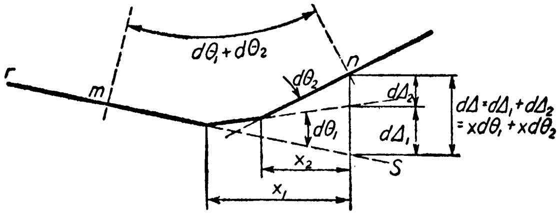


FIG. 17.

the axis of the beam is shown, it is not difficult to see that the angle between sections m and n will be the sum of the two rotations $d\theta_1$ and $d\theta_2$ while the displacement $d\Delta$ of point B from the axis rs will be the sum of $d\Delta_1$ and $d\Delta_2$, or

$$d\Delta = x_1\,d\theta_1 + x_2\,d\theta_2 \tag{15}$$

When this procedure is extended to include the deformation of all elements of the beam, it follows from the above discussion that the total relative rotation between any two sections m and n will be the summation of all the $d\theta$'s between them, or

$$\theta_{mn} = \int_m^n d\theta = \int_m^n \frac{M\,dx}{EI}$$

$$= \textbf{Area of } \frac{M}{EI} \textbf{ diagram between } m \textbf{ and } n \tag{16}$$

It will also be evident that the total displacement Δ of point B from the tangent rs will be the summation of the products $x\,d\theta$ for each element between the sections m and n, or

$$\Delta = \int_m^n x\,d\theta = \int_m^n \frac{M}{EI}\,x\,dx$$

$$= \textbf{Statical moment of } \frac{M}{EI} \textbf{ diagram between m and } n \textbf{ about } n \tag{17}$$

The geometrical relations that are expressed by equations 16 and 17 are designated as the *first* and *second theorems of area moments*, re-

spectively.* Special consideration should be given to the fact that the actual position of the axis of the deflected beam cannot be determined until the position of one section or tangent is known. For instance, if the beam in Fig. 18*a* has a constant cross section, the shape of the $\frac{M}{EI}$ diagram will be as shown in Fig. 18*c*. The value of the angle θ

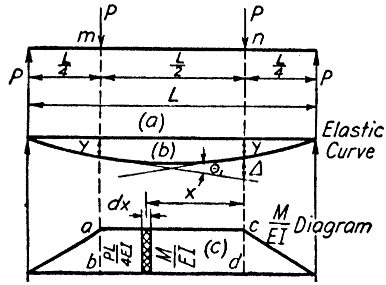


FIG. 18.

between the tangents to the elastic curve at sections m and n (Fig. 18*b*) is equal to the area of the $\frac{M}{EI}$ diagram between sections m and n, or the rectangle *abcd*. That is,

$$\theta_1 = \text{area of } \frac{M}{EI} \text{ diagram between } m \text{ and } n$$

or

$$\theta_1 = \int_n^m \frac{M\,dx}{EI} = \left(\frac{PL}{4EI}\right)\left(\frac{L}{2}\right) = \frac{PL^2}{8EI} = \frac{\text{lb in.}^2\,\text{in.}^2}{\text{lb in.}^4} = \text{radians}$$

However, the angle through which the tangent at m or n has actually rotated is not known until a further study is made.

The movement Δ of point n with respect to a tangent at m can be obtained from theorem II, equation 17, by taking the statical moment of the area of the $\frac{M}{EI}$ diagram between sections m and n (*abcd*) with respect to n (point where Δ is desired). This numerical operation gives

$$\Delta = \int_n^m \left(\frac{M}{EI}dx\right)(x) = \int_n^m x\,dA = \left(\frac{PL}{4EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{4}\right) = \frac{PL^3}{32EI}$$

It will be emphasized again that the actual displacement y is still unknown.

* These theorems were first presented by Charles E. Greene, University of Michigan (1873), and by Otto Mohr in Germany (about 1868).

For a beam with a fixed tangent, as at end A of beam AB, Fig. 19*a*, where the tangent is prevented from rotating by the support, the rotations θ and the displacements Δ will be the actual movements. From the $\dfrac{M}{EI}$ diagram, Fig. 19*b*, these quantities are obtained directly, that is

$$\theta_b = \text{Area of } \frac{M}{EI} \text{ diagram between } A \text{ and } B = \frac{Pl^2}{2EI}$$

$$\Delta_b = \text{Statical moment of } \frac{M}{EI} \text{ diagram about } B = \frac{Pl^3}{3EI}$$

From the above discussion it is apparent that the theorems of area moment can be used directly for the solution of *fixed-end beams*. Moreover, the reader should note that, if the value of EI is not constant, the $\dfrac{M}{EI}$ diagram will not take the same shape as the bending-moment diagram. In this case, the numerical solution is often simplified by substituting an *approximate summation* for the *exact integration*, a procedure that is illustrated later.

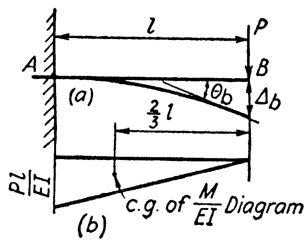


FIG. 19.

14. Conjugate-Beam Method (Actual Displacements in Beams). The actual displacements in a beam with two non-yielding simple supports are easily calculated by a further development of the slope and moment-area relationship that is commonly designated as the *conjugate-beam method* (Ref. 2·10) or the method of elastic weights (Ref. 2·4).

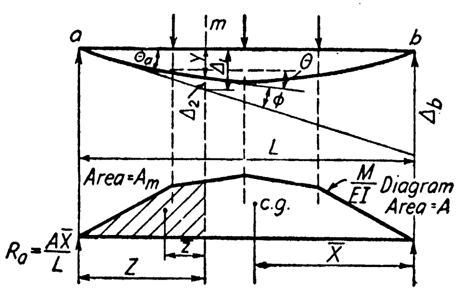


FIG. 20.

This convenient mathematical analogy is readily derived by the geometrical relations illustrated in Fig. 20. If we let A equal the area of the $\dfrac{M}{EI}$ diagram for the entire span ab, and if $\bar{x}$ is the distance from b

to the centroid of this area, then, by equation 17, Δ_b is equal to $A\bar{x}$, and the end rotation θ_a (for small angles) is therefore equal to $\frac{\Delta_b}{L}$, or

$$\theta_a = \frac{A\bar{x}}{L} \tag{18}$$

The actual rotation θ of the tangent at any section m will be the end rotation θ_a minus ϕ, the relative rotation between sections a and m which is equal to the shaded area A_m between the two sections. Therefore,

$$\theta = \theta_a - \phi = \frac{A\bar{x}}{L} - A_m \tag{19}$$

If we regard the $\frac{M}{EI}$ diagram as a weight on a beam of span L with supports at a and b, which will be called the *conjugate-beam,* then $\frac{A\bar{x}}{L}$, or θ_a, will equal the left reaction of this beam, and the rotation θ is equal to the reaction minus the load A_m, or equals the shear in the conjugate beam at that section.

Also, the actual displacement y at the section m is equal to $\Delta_1 - \Delta_2$, or

$$y = \Delta_1 - \Delta_2 = (\theta_a)(z) - A_m\bar{z} = \frac{A\bar{x}}{L}z - A_m\bar{z} \tag{20}$$

Equation 20 states that the ordinate y to the elastic curve of a beam at any section m is numerically equal to the moment of the reaction of the conjugate beam $\frac{A\bar{x}}{L}$ about section m minus the moment of the $\frac{M}{EI}$ diagram between the reaction and the section about section m. Therefore, y equals the bending moment in the conjugate beam.

The above analogy can be conveniently summarized as follows. To obtain the rotations and displacements for any beam with two non-yielding supports a and b, construct a conjugate beam by using the $\frac{M}{EI}$ diagram as a load on a similar beam of span ab. Compute the value of the reactions, shear, and bending moment in the conjugate beam for the $\frac{M}{EI}$ loading. From equations 19 and 20, **the shear in the conjugate beam equals the rotation in the actual beam, and the bending moment in the conjugate beam equals the displacement in the actual beam.** This analogy holds for any continuous elastic curve between the two non-yielding supports a and b.

Example 6. The end rotations and maximum displacement in beam AB, Fig. 21, will be computed by means of the conjugate beam $A'B'$. The area A of the $\frac{M}{EI}$ diagram is

$$A = \frac{1}{2}\frac{20{,}000}{EI}15 = \frac{150{,}000}{EI}$$

$$R'_a = \theta_a = \frac{\left(\frac{150{,}000}{EI}\right)\left(\frac{20}{3}\right)}{15} = \frac{200{,}000}{3EI} \text{ radians}$$

$$R'_b = \theta_b = \frac{\left(\frac{150{,}000}{EI}\right)\left(\frac{25}{3}\right)}{15} = \frac{250{,}000}{3EI} \text{ radians}$$

The value of EI must be in pounds times feet squared.

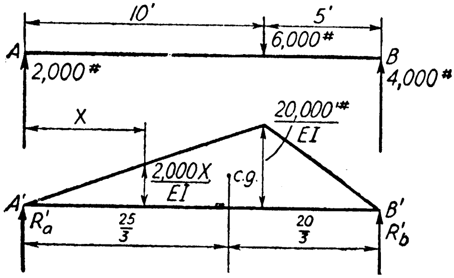


FIG. 21.

The maximum displacement in AB will occur where the slope is zero, that is, where the shear in the conjugate beam $A'B'$ is zero. The shear in the conjugate beam at any distance x from A' is

$$V' = R'_a - \left(\frac{1}{2}\right)(x)\left(\frac{2000x}{EI}\right) = \frac{200{,}000}{3EI} - \frac{1000x^2}{EI}$$

and, therefore, for zero shear

$$x^2 = 66.7 \qquad x = 8.15 \text{ ft}$$

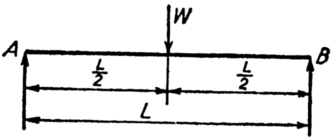
The displacement at this point is equal to the bending moment in the conjugate beam, or

$$y_{\max} = \left(\frac{200{,}000}{3EI}\right)(8.15) - \left(\frac{1}{2}\right)(8.15)\left(\frac{16{,}300}{EI}\right)\left(\frac{8.15}{3}\right) = \frac{362{,}890}{EI} \text{ ft}$$

PROBLEMS

16. Determine the displacement under W and the rotation of the tangent at A in terms of EI by the conjugate-beam method.

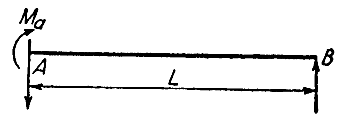
Ans. $\theta_a = \dfrac{WL^2}{16EI}$.



PROBLEM 16.

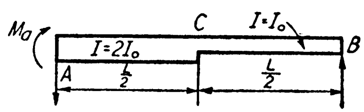
17. Determine the rotations of the tangents at A and B in terms of EI.

Ans. $\theta_a = \dfrac{M_aL}{3EI}$; $\theta_b = \dfrac{M_aL}{6EI}$.



PROBLEM 17.

18. Determine the rotations of the tangents at A and B in terms of M_a, L, and EI_0.



PROBLEM 18.

19. Calculate the rotations θ_a and θ_b and the displacement Δ_c for a vertical load P at the center of the beam in Problem 18 if M_a is removed.

15. Calculation of Redundant Forces from Specified Beam Displacements. The numerical calculation of displacements in the preceding examples have been made for statically determinate beams. The elastic curve of a beam can be completely determined by the above methods for any known force system. However, there is no reason why the displacements cannot be expressed in terms of any number of forces, known or unknown, by means of equations 16, 17, 19, and 20. The development and use of such equations are treated in more detail in succeeding chapters, but a few applications can be discussed conveniently at this time. If certain strain conditions, as discussed in Chapter I, are known, one redundant force can be determined for each specified strain condition. These strain conditions can be expressed in terms of either rotations or translations.

When drawing the $\frac{M}{EI}$ diagram for general force systems, a convenient arrangement is to separate the known loads and their reactions from the redundant forces and their reactions. This resolution of forces is permissible if the principle of superposition can be applied. Any displacement is then expressed in terms of both the given loading and the unknown forces, from which the unknown forces can be calculated so as to satisfy any known strain conditions. The numerical procedure is best explained by the following examples.

Example 7. The fixed-end moment M_{Fab} for the beam *abc*, Fig. 22*a*, will be determined for a uniform load of 2 kips per foot (EI = constant). As previously defined, a fixed-end moment is an end couple that will prevent rotation of the cross section at *a*.

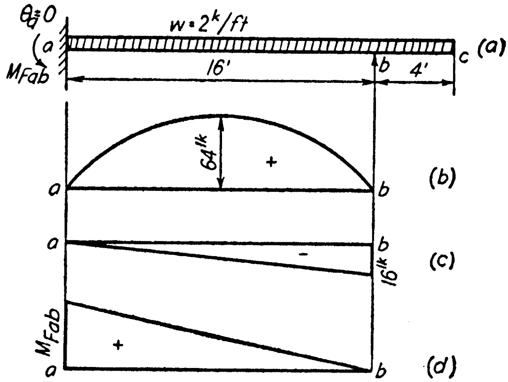


FIG. 22.

For convenience the $\frac{M}{EI}$ diagram for the span *ab* is drawn in three parts: (1) for the uniform load on the span *ab*, Fig. 22*b;* (2) for the uniform load on the cantilever portion *bc*, Fig. 22*c;* (3) for the fixed-end moment M_{Fab} which is assumed positive, Fig. 22*d*. These three diagrams constitute the load on the conjugate beam, and, if the tangent at *a* is to remain horizontal ($\theta_a = 0$), the shear in the conjugate beam from this loading must be zero, or, taking the statical moments of the $\frac{M}{EI}$ diagrams about point *b*,

$$M_b = \left(\frac{2}{3}\right)\left(\frac{64}{EI}\right)(16)(8) - \left(\frac{1}{2}\right)\frac{16}{EI}(16)\left(\frac{16}{3}\right) + \frac{1}{2}\frac{M_{Fab}}{EI}(16)\left(\frac{32}{3}\right) = 0$$

from which

$$M_{Fab} = -56 \text{ ft-kips}$$

The negative sign shows that the assumed direction is wrong.

The reaction R_b is now known as

$$R_b = 16 + 9 - \tfrac{56}{16} = 21.5 \text{ kips}$$

Example 8. The fixed-end moments M_{Fab} and M_{Fba}, Fig. 23*a*, will be calculated for the concentrated load of 10 kips at the center of the span and for the variation in I as shown. Again the $\dfrac{M}{EI}$ diagram is drawn for three different force systems: (1) for the concentrated load of 10 kips, Fig. 23*b;* (2) for the fixed-end moment M_{Fab}, assumed clockwise, Fig. 23*c;* (3) for the fixed-end moment M_{Fba} assumed clockwise,

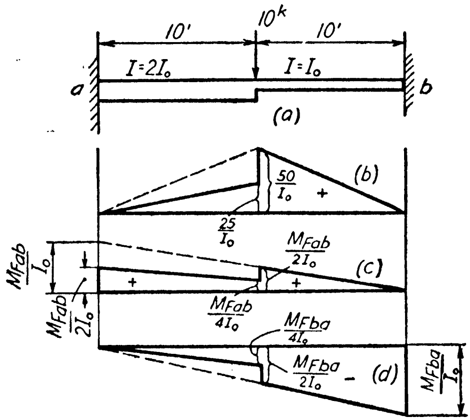


FIG. 23.

Fig. 23*d*. In this problem the shear at both ends of the conjugate beam must be zero ($\theta_a = \theta_b = 0$). Consequently, the statical moment of the combined $\dfrac{M}{EI}$ diagram about both a and b must be zero,

$$\Sigma M_b = 0 \qquad \left(\frac{1}{EI_0} \text{ is omitted from all terms}\right)$$

$$125\left(\frac{40}{3}\right) + 250\left(\frac{20}{3}\right) + M_{Fab}\left[1.25\left(\frac{50}{3}\right) + (2.5)(15) + \frac{(2.5)(20)}{3}\right]$$
$$-M_{Fba}\left[(1.25)\left(\frac{40}{3}\right) + (5)(5) + (2.5)\left(\frac{10}{3}\right)\right] = 0$$

$$\Sigma M_a = 0$$

$$125\left(\frac{20}{3}\right) + (250)\left(\frac{40}{3}\right) + M_{Fab}\left[(1.25)\left(\frac{10}{3}\right) + (2.5)(5) + (2.5)\left(\frac{40}{3}\right)\right]$$
$$-M_{Fba}\left[(1.25)\left(\frac{20}{3}\right) + (5)(15) + (2.5)\left(\frac{50}{3}\right)\right] = 0$$

or

$$22.5M_{Fab} - 15.0M_{Fba} = -1000$$

$$15.0M_{Fab} - 37.5M_{Fba} = -1250$$

from which

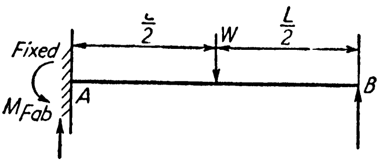
$$M_{Fab} = -30.3 \text{ ft-kips} \qquad M_{Fba} = 21.2 \text{ ft-kips}$$

The negative sign for M_{Fab} shows that it is counterclockwise instead of clockwise as assumed.

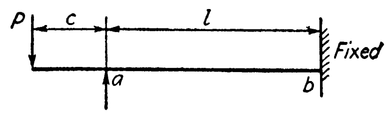
PROBLEMS

20. Determine the fixed-end moment M_{Fab} (EI = constant).

Ans. $M_{Fab} = -\frac{3}{16} WL.$



PROBLEM 20.



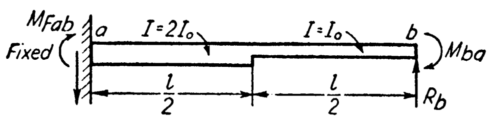
PROBLEM 21.

21. Calculate the value of M_{Fba} for a load P as shown. Beam is simply supported at a and fixed at b. EI is constant.

Ans. $M_{Fba} = -\frac{Pc}{2}.$

22. Calculate the value of M_{Fab} and the rotation of the tangent at b for a moment M_{ba} applied at end b and variation in I as shown.

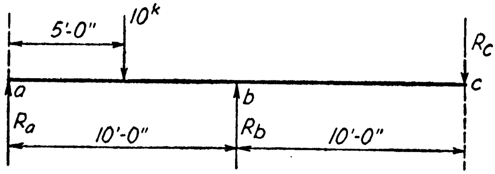
Ans. $M_{Fab} = \frac{2}{3} M_{ba}$; $\theta_b = 0.229 \frac{M_{ba}l}{EI_0}.$



PROBLEM 22.

23. Determine the value of R_b so that points a, b, c will remain on a straight line (EI constant). (Suggestion: Consider a single span ac acted upon by two loads, the applied load of 10 kips and the redundant force R_b.)

Ans. $R_b = 6.88$ kips.



PROBLEM 23.

24. Solve Problem 23 by considering the moment M_b as the redundant force and continuity of the elastic curve over R_b as the known strain condition to be satisfied.

16. Williot Diagram (Relative Displacements in Trusses). In many structures, particularly trusses, the members are subjected to large axial stresses which cause a change in length of the member equal to $\frac{SL}{AE}$, where $\frac{S}{A}$ is the average unit stress. As the allowable average stress for structural steel will not usually exceed 20,000 lb per in.^{2}, the total deformation will be less than $\frac{L}{1500}$ and consequently it is practically impossible to show the change in length of the members to the

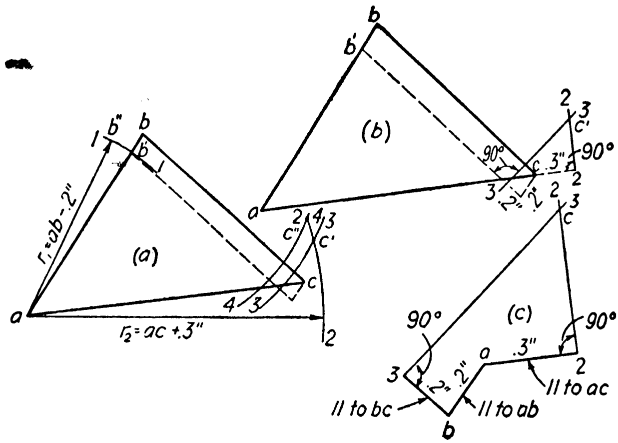


FIG. 24.

same scale as the truss itself is drawn. To avoid this difficulty, a diagram that uses only the *change in length of the members* due to the axial stress was devised by Williot, a French engineer, to show graphically the relative motion of all points with respect to one point and a reference axis. This information is often used directly in the solution of special problems, such as the calculation of the secondary stresses in large trusses, but in addition, when combined with a rotation diagram, it provides an extremely practical method for the determination of the actual movement of all joints. The construction of the Williot diagram will therefore be discussed in detail.

If three points, a, b, and c, are connected by any three members as in Fig. 24a, and if ab and bc both shorten 0.2 in. while ac elongates 0.3 in., the new position of c can be determined if the positions of a and b are known. For instance, if point a remains in position, then point b must be somewhere on arc 1–1 whose radius is (ab − 0.2 in.) (not to scale) while c will be somewhere on arc 2–2 whose radius is (ac + 0.3 in.).

However, before the position of c on arc 2–2 can be determined, the position of b on arc 1–1 must be known. Thus, if b moves to b', then c must be at c', the intersection of arcs 2 and 3 (radius bc − 0.2 in.), whereas, if b moves to b'', then c must be at c'', the intersection of arcs 2 and 4. Now, if we assume that b moves to b', then, since the deformation has been shown to be extremely small as compared with the length of the member, the same movements can be obtained with sufficient accuracy by using perpendiculars as shown in Fig. 24b instead of arcs. Moreover, it is now apparent that the movements represented by these

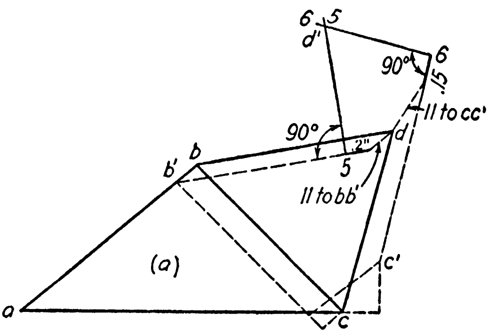


FIG. 25a.

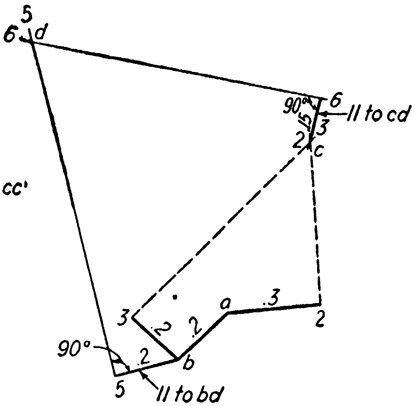


FIG. 25b.

perpendiculars can be drawn separately to a large scale as in Fig. 24c, which gives the motion of c relative to a and b. After the position of point c with respect to a and b has been obtained, the same procedure can be followed to obtain the motion of another point d (Fig. 25a) with respect to b and c. Both this movement and the preceding one are shown in Fig. 25a. The member bd shortens 0.2 in., which moves d toward b by that amount, while the member cd elongates 0.15 in., which moves d away from c by that amount. The intersection of the perpendiculars 5 and 6 will therefore give the movement of d with respect to b, c, and a. The construction in Fig. 25b is so arranged that the displacement vectors need not be duplicated for each joint but are drawn only once. In this respect, the diagram is similar to a stress diagram for a truss, which combines force polygons for each joint without repeating the stress vectors.

Example 9. The Williot diagram in Fig. 26 shows the relative displacements for the truss in Fig. 9a (see data in Table 1). The diagram was started by assuming that point a does not move, which is correct,

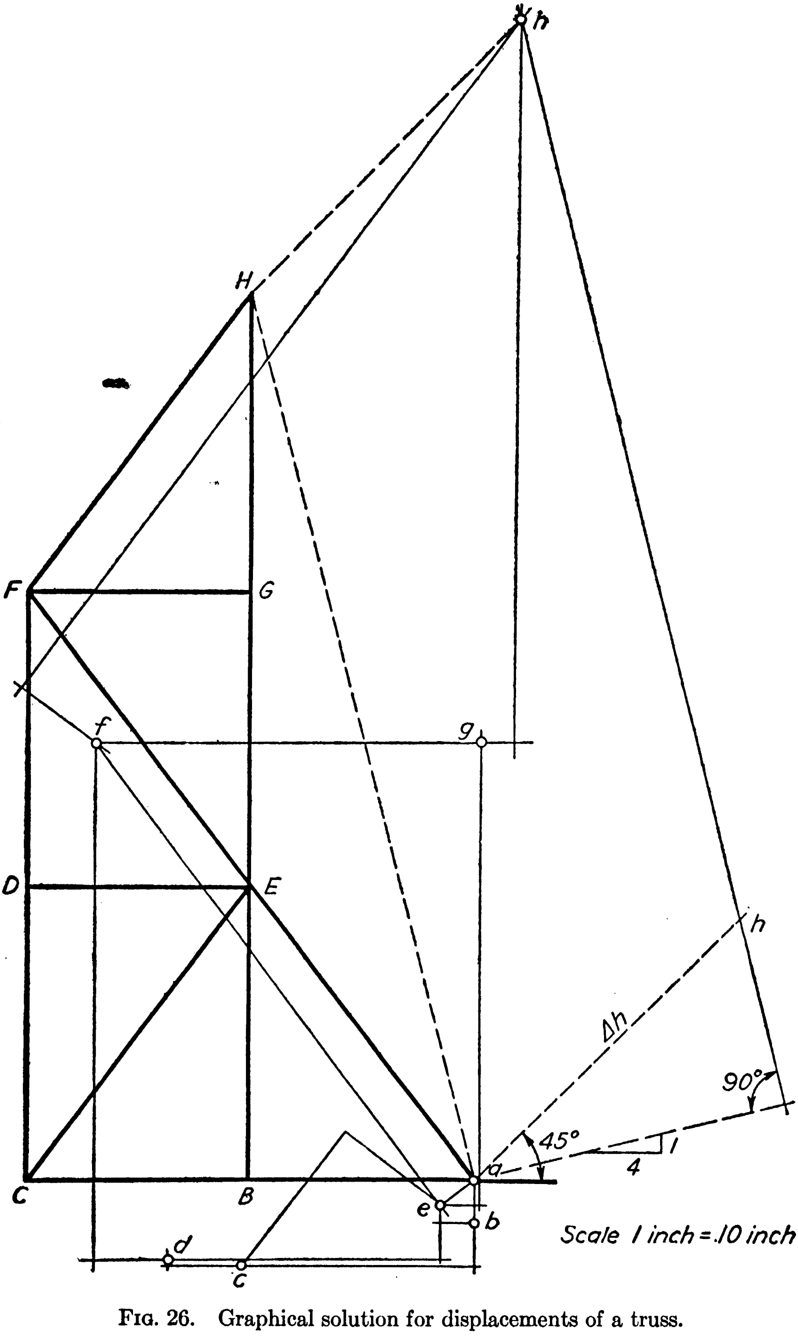


FIG. 26. Graphical solution for displacements of a truss.

and that point b moves vertically toward a, which is incorrect. However, this incorrect assumption, that the member ab does not rotate, does not affect the relative displacements of the joints, and, as we shall see later, the error can be corrected so as to give the true displacements. By starting with known positions of a and b, the movement of the joint e can be obtained in the manner just described, and then joints c, d, f, g, and h, in the order named. Point e is peculiar in that the end of the vector representing the movement ae intersects on the perpendicular to the vector be. All graphical constructions should be studied until each step is thoroughly understood.

17. Rotation Diagrams (Actual Displacements in Trusses). The Williot diagram determines the actual displacement of the joints only

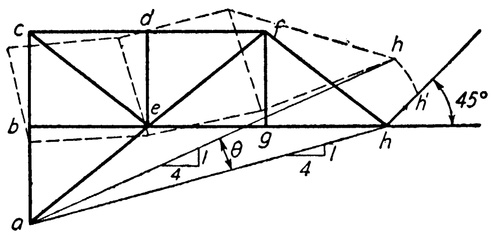


FIG. 27*a*.

if the two reference points that start the construction have their correct position with respect to each other. Sometimes this condition can be realized for a particular loading, but more frequently it is necessary to start with a known fixed point and some reference axis that is assumed not to rotate. This assumption was made in constructing the Williot diagram in Fig. 26, where a is the fixed point and ab the reference axis. If the deflected position of the truss is drawn by using the relative displacements from the Williot diagram, a diagram like Fig. 27*a* is obtained, in which the displacements are, of course, greatly exaggerated. Since reaction h is now some distance off the support, it is apparent that the assumption that the member ab does not rotate is incorrect, and that the entire structure must be rotated about point a until point h comes back to the support. As all movements are small, the displacement hh' can be taken perpendicular to ah, and the magnitude of the displacement is the distance hh' in Fig. 26. The movements of the other points due to this rotation θ are shown in Fig. 27*b*, where it can be seen that each point moves perpendicular to a line connecting it with point a and the magnitude of the motion is proportional to the distance from a and the angle θ. The magnitude of the rotation θ is obtained from the distance hh'. Professor Mohr was the first to point

out that the movement of all points due to rotation of the structure can be obtained graphically, if one displacement such as hh' in Fig. 27*b* is known, by drawing *a diagram of the truss rotated 90°* as in Fig. 27*c*. For example, in Fig. 27*b*, if the distance ah is made to represent the distance hh' to some scale, then by proportion ad represents dd', ac represents cc', etc. If this displacement diagram is now rotated until the member ah is in the direction of hh' (Fig. 27*c*), that is rotated 90°,

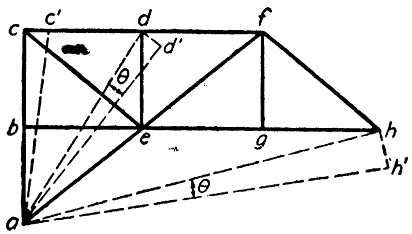


FIG. 27*b*.

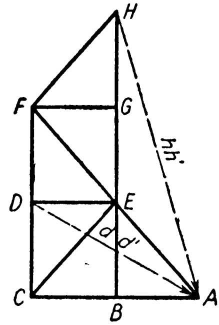


FIG. 27*c*.

the displacement hh' will be given both in magnitude and direction by AH and the rotated figure becomes a vector diagram for all displacements due to rotation.

If the Mohr rotation diagram is drawn to the same scale as the Williot diagram and superimposed upon it so that the fixed points, such as point a in Fig. 26, coincide, then the actual displacement of any point is the vectorial sum of the displacement vectors of the two diagrams. As an illustration, in Fig. 26, the relative motion of point d is given by a vector ad and the movement due to rotation is given by a vector Da, consequently the actual displacement is the resultant Dd. Similarly, Bb, Cc, Ee, etc., represent the actual displacements of points b, c, e, respectively.

If the Williot diagram is not started from any fixed point but with a known axis, a correction will be necessary only for translation of the structure, but, if neither a fixed point nor a fixed axis is used, a correction must be made for both translation and rotation. Such corrections are not difficult to execute, and they often provide some interesting graphical solutions. In general, either a fixed point or a fixed axis is taken in starting the Williot diagram, but at times another point may be more convenient.

18. Relative Displacements in Frames. A Williot diagram can also be used to obtain the relative displacements of the ends of members

that undergo considerable flexure but have small axial stresses. This is the condition that usually exists in rigid-frame structures, and a study of such displacements is an important factor in their analysis. For example, if the frame *abcd* (Fig. 28*a*) is moved to the right by the load *P*, the displacements of the joints will be largely due to the internal strains that are caused by bending moments, instead of axial stress as in a truss. If the change in length of the members is neglected,

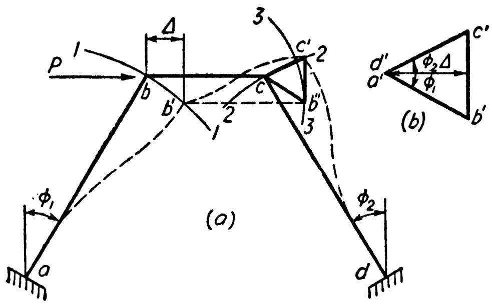


FIG. 28. Relative displacements in a quadrangular frame.

point *b* must be somewhere on arc 1–1 which has the fixed point *a* for a center and *ab* as radius. Also, point *c* must be somewhere on arc 2 whose radius is *cd* and arc 3 whose radius is *bc* and center *b'*. As the rotations of the various members are small, the arcs can be replaced by perpendiculars as for trusses. From the construction it is apparent that the triangle *cc'b''* gives the movement of *b* with respect to *a*, and *c* with respect to *b* and *d*. If this diagram is drawn out separately, as in Fig. 28*b*, it is identical in construction with a Williot diagram in which the elongations are all zero. Thus, *a'b'* is perpendicular to *ab*, *c'd'* to *cd*, *b'c'* to *bc*. All displacements can be expressed in terms of Δ, the horizontal movement, as follows:

$$
\begin{aligned}
a'b' &= bb' = \Delta \sec \phi_1 \\
c'd' &= cc' = \Delta \sec \phi_2 \qquad (21) \\
b'c' &= \Delta (\tan \phi_1 + \tan \phi_2)
\end{aligned}
$$

The value of Δ can be ascertained by methods that will be explained later.

The same procedure can be followed for the gable frame in Fig. 29*a*, in which *b* and *d* must move horizontally to *b'* and *d'*, respectively. Point *c* will then move to *c'*, which is on the intersection of arcs 1–1 and 2–2. If the arcs are replaced by perpendiculars, the movement of all joints can be shown by the displacement diagram of Fig. 29*b*,

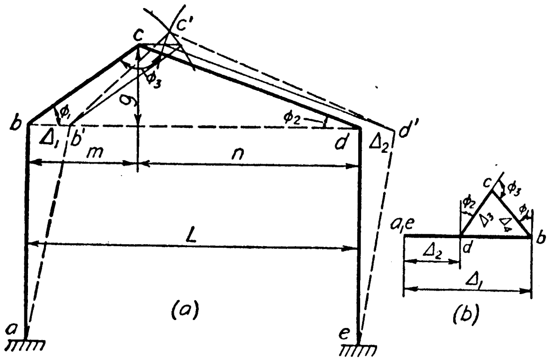


FIG. 29. Relative displacements in a gable frame.

which is simply the construction at c in Fig. 29a to a larger scale. From the geometrical relations of the displacement diagram

$$\frac{\Delta_4}{\sin (90° - \phi_2)} = \frac{\Delta_1 - \Delta_2}{\sin (180° - \phi_3)}$$

or

$$\Delta_4 = (\Delta_1 - \Delta_2) \frac{\cos \phi_2}{\sin (180° - \phi_3)}$$

but

$$\cos \phi_2 = \frac{n}{cd}$$

and

$$\sin (180° - \phi_3) = \frac{L \sin \phi_1}{cd} = \frac{Lg}{(cd)(bc)}$$

from which the rotation of member bc is equal to

$$\frac{\Delta_4}{bc} = (\Delta_1 - \Delta_2) \frac{n}{gL} \tag{22}$$

Similarly the rotation of cd which is equal to $\frac{\Delta_3}{cd}$ is obtained from the relation:

$$\frac{\Delta_3}{\sin (90° - \phi_1)} = \frac{\Delta_1 - \Delta_2}{\sin (180° - \phi_3)}$$

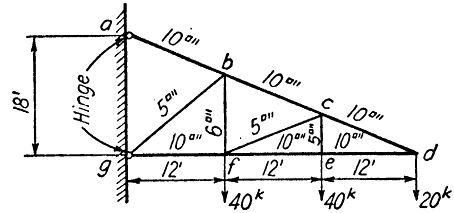
which, by the relations already established, is equal to

$$\frac{\Delta_3}{cd} = (\Delta_1 - \Delta_2) \frac{m}{gL} \tag{23}$$

In the analysis of such frames, it is therefore necessary to consider only the two independent displacements, Δ_1 and Δ_2.

PROBLEMS

25. Determine the displacements of all joints by means of a Williot diagram.

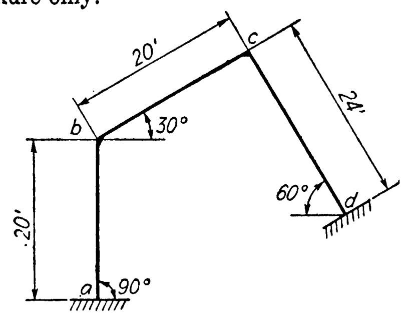


PROBLEM 25.

26. Solve Problems 7, 8, and 9 by means of a Williot diagram combined with a Mohr rotation diagram.

27. Solve Problem 10 by means of a Williot diagram. Note that member U_2L_2 does not rotate.

28. If joint b moves 10 units horizontally, calculate the rotations of members bc and cd due to flexure only.



PROBLEM 28.

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CHAPTER III

CONTINUOUS BEAMS AND FRAMES WITH STRAIGHT PRISMATIC MEMBERS

Derivation and Application of Slope-Deflection Equations

19. Definitions and Assumptions. The term continuity is applied to structures when the members are so connected that moments as well as shears and thrusts are transmitted from one member or unit to another. The building frame in Fig. 30 illustrates this type of framing, which has been used in reinforced-concrete construction for many years and is now frequently employed in steel structures. When there is practically no deformation in the connection itself, the continuous frame is often called a rigid frame. The characteristic action of such a structure is **the restraint or assistance that the columns give to the beams when the beams are subjected to vertical loads and the restraint that the beams give to the columns when the frame is subjected to horizontal loads,** such as those caused by wind or earthquake action.

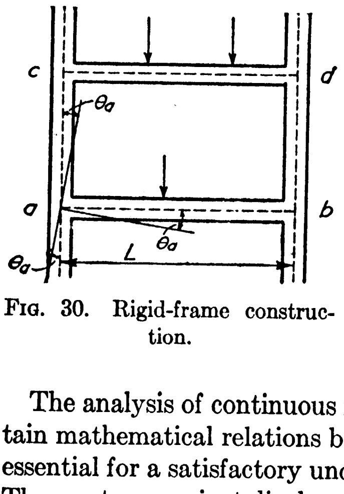


FIG. 30. Rigid-frame construction.

The analysis of continuous frames will be initiated by a study of certain mathematical relations between forces and displacements that are essential for a satisfactory understanding of recent analytical methods. The most convenient displacements to use in the solution of many continuous frame structures are the rotations θ and the translations Δ of the various joints. The term joint denotes the material at the intersection of several members that is common to all. It cannot be described exactly. A common example is given in Fig. 30, in which each member of the frame is considered to be of constant cross section up to the intersection of the dotted lines that represent the axes of the members. Such an assumption will reduce the joint to a point, a conception that is useful for analytical purposes but which must be modified in the final design. If the elastic curve of each member is assumed

to be continuous to the intersection of the axes and no deformation takes place within the joint, then the rotation of the end tangent must be the same for each member, as for example θ_a in Fig. 30. This rotation can also be described as the joint rotation. Any position of the frame can be defined in terms of the displacements of the joints and the elastic curves of the members.

The coplanar end forces acting upon any member of the frame, such as *ab*, are shown in Fig. 31. As there are three forces acting at the end of each member, the number of forces that must be considered, if used as unknowns, will increase rapidly as the number of members increases. On the other hand, if the rotations and displacements of the joints are used as unknowns, the number increases only with the number of joints. Another advantage already mentioned is that the necessary strain conditions at the joints can be automatically satisfied. That is, by using but one θ and usually one Δ value at each joint, the condition of perfect continuity is satisfied. Before the strain conditions can be expressed in algebraic form, however, it is necessary to know the mathematical relations that exist between the end forces and the rotations and displacements at the ends of the members. Several methods for solving these problems, as well as a number of examples, were explained in Chapter II, and therefore the student should be able to provide most of the derivations for himself. The most important relations will now be discussed.

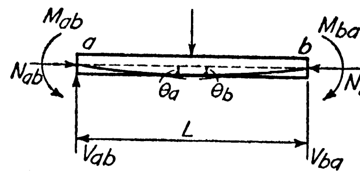


FIG. 31.

20. End Moments Expressed in Terms of End Rotations and Translations. Two apparently different derivations are available for expressing the mathematical relations that exist between the forces acting on any member of a continuous frame, like that in Fig. 31, and the rotations θ and displacements Δ of the end sections. Actually, the only difference is in the nature of the different force systems into which the force system is resolved for analytical purposes. The four separate force systems of Fig. 32*a*, *b*, *c*, and *d*, indicate one method of resolving the forces in Fig. 31. The forces in Fig. 32*a* prevent rotation of the end tangents when the transverse loads are applied. The force system of Fig. 32*b* will rotate the tangent at *a* through an angle θ_a while the tangent at *b* remains fixed, and the forces in Fig. 32*c* will rotate the tangent at *b* through an angle θ_b while the tangent at *a* remains fixed. The end forces in Fig. 32*d* hold both end tangents fixed when the two ends undergo a relative transverse displacement Δ. By adding these

force systems together any desired set of displacements θ and Δ can be obtained, or by assigning numerical values to θ and Δ the forces can be calculated. For any given structure and loading, only one set of displacements will satisfy equilibrium conditions because the number of displacements is just equal to the number of equilibrium equations.

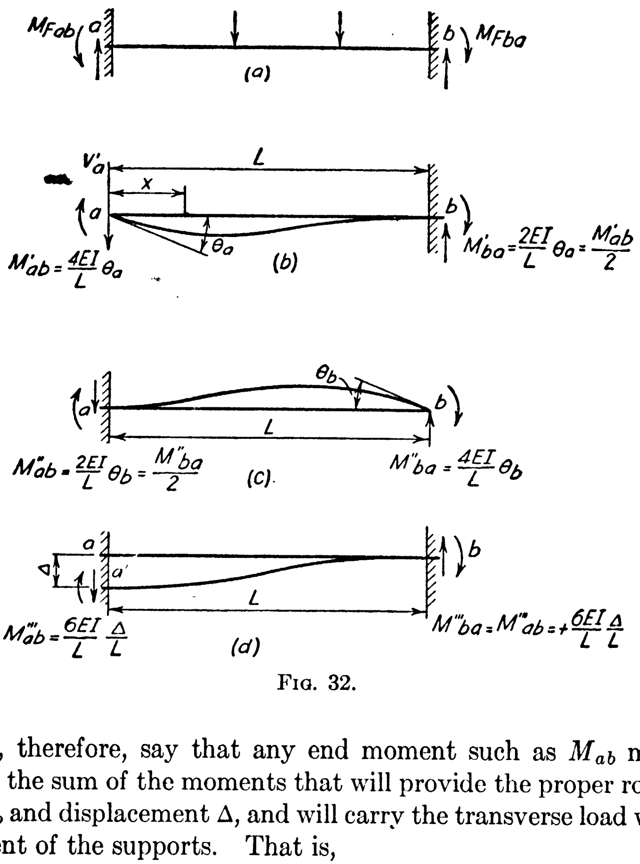


FIG. 32.

We can, therefore, say that any end moment such as M_{ab} must be equal to the sum of the moments that will provide the proper rotations θ_a and θ_b and displacement Δ, and will carry the transverse load without movement of the supports. That is,

$$M_{ab} = M_{Fab} + M'_{ab} + M''_{ab} + M'''_{ab} \qquad (24a)$$

$$M_{ba} = M_{Fba} + M'_{ba} + M''_{ba} + M'''_{ba} \qquad (24b)$$

Each of the above four problems can be solved by means of the *conjugate-beam method* or *the theorems of area moments.* The reader should refer to Articles 13, 14, and 15 and check the numerical values given in Fig. 32 for himself.

If the moments are now expressed in terms of the rotations and translations at the ends of the member, equations 24*a* and 24*b* become

$$M_{ab} = M_{Fab} + \frac{EI}{L}\left[4\theta_a + 2\theta_b \pm 6\frac{\Delta}{L}\right] \tag{25a}$$

$$M_{ba} = M_{Fba} + \frac{EI}{L}\left[2\theta_a + 4\theta_b \pm 6\frac{\Delta}{L}\right] \tag{25b}$$

In these equations the numerical values of the fixed-end moments M_{Fab} and M_{Fba} should be used with their *proper signs: positive* when acting *clockwise* on the member and *negative* when acting *counterclockwise* on the member. The unknown angles θ should be assumed *positive* or *clockwise*, while the direction of Δ can be taken either positive or negative, if unknown. The sign of the fixed-end moments due to Δ, $\left(\frac{6EI\Delta}{L^2}\right)$, must be consistent with the direction of Δ.

When one end of a member is hinged, such as b, the moment M_{ba} equals zero and the angle θ_b can be eliminated from equation 25*a*. This simplification gives the following equation:

$$M_{ab} = M_{Fab} - \frac{M_{Fba}}{2} + 3\frac{EI}{L}\left(\theta_a \pm \frac{\Delta}{L}\right) \tag{26a}$$

When M_{ab} equals zero, a similar equation is obtained for M_{ba}

$$M_{ba} = M_{Fba} - \frac{M_{Fab}}{2} + 3\frac{EI}{L}\left(\theta_b \pm \frac{\Delta}{L}\right) \tag{26b}$$

As the fixed-end moments M_{Fab} and M_{Fba} are usually of opposite sign these terms will actually be added when the numerical values are substituted in equations 26*a* and 26*b* with their proper signs.

Equations 25*a* and 25*b* can also be obtained by using the force systems shown in Fig. 33*a*, *b*, *c*, and *d*, which, when added together, give the actual forces acting on any member. For this combination of force systems it is necessary that the resultant angle θ be the sum of the angles caused by the different force systems, that is

$$\theta_a = \alpha_a + \theta_a' - \theta_a'' - \frac{\Delta}{L} \tag{27a}$$

$$\theta_b = -\alpha_b - \theta_b' + \theta_b'' - \frac{\Delta}{L} \tag{27b}$$

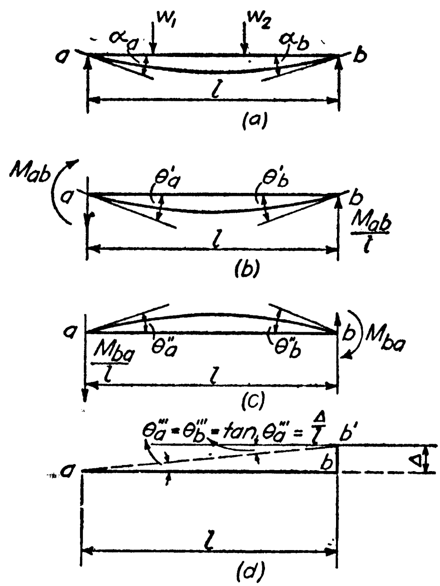


FIG. 33.

When the values of θ'_a, θ''_a, θ'_b, and θ''_b are expressed in terms of M_{ab} and M_{ba} (see conjugate beam, Article 14), equations 27*a* and 27*b* become

$$\theta_a = \alpha_a + \frac{M_{ab}L}{3EI} - \frac{M_{ba}L}{6EI} - \frac{\Delta}{L} \tag{27c}$$

$$\theta_b = -\alpha_b - \frac{M_{ab}L}{6EI} + \frac{M_{ba}L}{3EI} - \frac{\Delta}{L} \tag{27d}$$

Solving these equations for M_{ab} and M_{ba} gives

$$M_{ab} = \frac{EI}{L}\left[4\theta_a + 2\theta_b + 6\frac{\Delta}{L} - 4\alpha_a + 2\alpha_b\right] \tag{28a}$$

$$M_{ba} = \frac{EI}{L}\left[2\theta_a + 4\theta_b + 6\frac{\Delta}{L} - 2\alpha_a + 4\alpha_b\right] \tag{28b}$$

Equations 25 and 28 are identical since

$$M_{Fab} = -\frac{EI}{L}[4\alpha_a - 2\alpha_b] \tag{29a}$$

$$M_{Fba} = -\frac{EI}{L}[2\alpha_a - 4\alpha_b] \tag{29b}$$

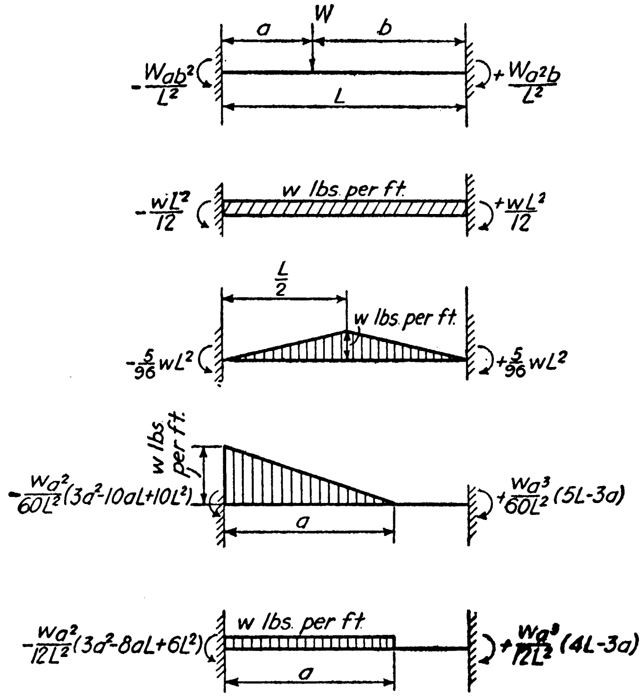


FIG. 34. Fixed-end moments for beams with constant EI.

As the angles α_a and α_b due to the applied loading are easily computed by the conjugate-beam method, equations 29*a* and 29*b* provide a convenient means for obtaining the *fixed-end moments.* Figure 34 gives values of the fixed-end moments for beams with constant cross section for ordinary types of loading.

21. Equilibrium Conditions. After the unknown end moments that act upon the various members have been expressed in terms of the rotations and translations of the joints, it is then necessary to consider the equilibrium requirements of the various parts of the structure. In general, the equilibrium conditions for each member and the structure as a whole must be satisfied and, further, the resultant of the external and internal forces acting upon any joint must be zero. It is therefore important that both correct and sufficient conditions of equilibrium be established.

To illustrate this statement, let us consider the end forces that act upon the members of the frame shown in Fig. 35 and their relation to the equilibrium of the structure. All end moments will be assumed positive or clockwise, while the direction of the H and V forces can be

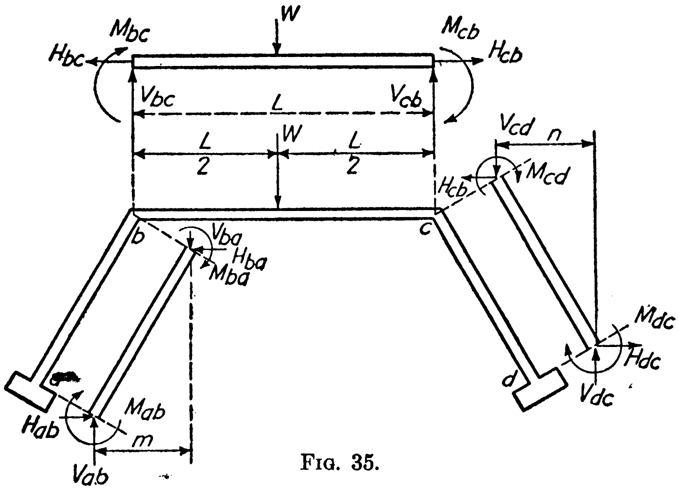


FIG. 35.

assumed in any convenient direction if the signs are consistent throughout. The conditions of equilibrium then require that the following equations be satisfied:

$$
\begin{array}{llll}
(a) & M_{ba} + M_{bc} = 0 & (d) & H_{ab} + H_{dc} = 0 \\
(b) & M_{cb} + M_{cd} = 0 & (e) & V_{ab} - V_{ba} = 0 \\
(c) & V_{ab} + V_{dc} - W = 0 & (f) & V_{dc} - V_{cd} = 0
\end{array}
\qquad (30)
$$

A study of the forces acting on each member will show that only one set of end moments will satisfy equations 30*a*, *b*, and *d* and, further, that any set that satisfies these equations can be used in the solution of equations 30*c*, *e*, and *f*. Therefore, the magnitude and direction of the end moments should be calculated from equations 30*a*, *b*, and *d*. The first two equations, 30*a* and *b*, are expressed directly in terms of the end moments, and equation 30*d* can be transformed into the same quantities by means of the following relations:

$$
\begin{aligned}
V_{ab} &= V_{bc} = \frac{W}{2} - \frac{M_{bc} + M_{cb}}{L} \\
H_{ab} &= \frac{M_{ab} + M_{ba} + V_{ab}m}{h} \\
V_{cb} &= V_{dc} = \frac{W}{2} + \frac{M_{bc} + M_{cb}}{L} \\
H_{dc} &= \frac{M_{cd} + M_{dc} - V_{dc}n}{h}
\end{aligned}
\qquad (31)
$$

When the above quantities are substituted in equation 30d there will be three independent equations each in terms of the unknown end moments. As all end moments can be expressed in terms of θ_b, θ_c, and Δ, the rotations and horizontal displacement of the joints b and c, by means of equations 25 or 26, there are sufficient equations for calculating these displacements. In the above equations h is the vertical projection of members ab and cd. The use of the slope-deflection equations in determining the end moments in certain types of rigid frame structures will now be illustrated by several numerical examples.

Example 10. A structure formed by three members whose axes are represented by ab, bc, and bd (Fig. 36) is so constructed that the end

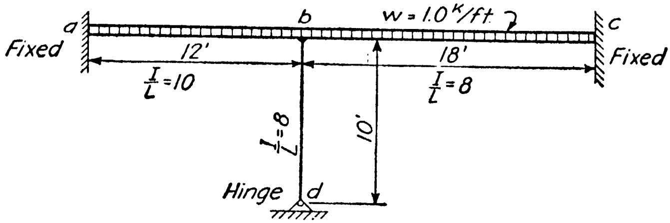


FIG. 36.

tangents at a and c are fixed in position ($\theta_a = \theta_c = 0$) while the tangent at d is free to rotate (hinged at d). The members are rigidly connected at b, that is, the tangent to each member at the joint b rotates through the same angle θ_b, thus maintaining a 90° angle between the tangents. No translation is considered.

The end moments acting on each member can be expressed in terms of the rotation of the joints, which in this particular case is only θ_b, by equations 25 or 26. For convenience in writing the equations, let θ_b equal E times the true angular rotation of b. Then, from equation 25,

$$M_{ab} = (2)(10)\theta_b + M_{Fab}$$

but

$$M_{Fab} = -\frac{wL^2}{12} = -\frac{(1\text{ kip})(12)^2}{12} = -12 \text{ ft-kips (counterclockwise)}$$

therefore,

$$\begin{aligned} M_{ab} &= 20\theta_b - 12 \\ M_{ba} &= (4)(10)\theta_b + M_{Fba} = 40\theta_b + 12.0 \\ M_{bc} &= (4)(8)(\theta_b) + M_{Fbc} = 32\theta_b - 27.0 \\ M_{cb} &= (2)(8)\theta_b + M_{Fcb} = 16\theta_b + 27.0 \end{aligned} \tag{32}$$

by equation 26, $M_{bd} = (3)(8)\theta_b = 24\theta_b$.

The above equations are dependent upon continuity of the members at joint b. Moreover, these moments must also satisfy the equilibrium condition for joint b, that is

$$M_{ba} + M_{bc} + M_{bd} = 0 \tag{33}$$

If the values in equation 32 are substituted in equation 33, we obtain:

$$40\theta_b + 12.0 + 32\theta_b - 27.0 + 24\theta_b = 0$$

$$96\theta_b = 15.0$$

$$\theta_b = 0.1563 \text{ (actually } E\theta)$$

When this value of θ_b is substituted back in equation 32 for the end couples, the following values are obtained:

$$M_{ab} = -8.87 \text{ ft-kips} \qquad M_{cb} = 29.5 \text{ ft-kips}$$

$$M_{ba} = 18.25 \text{ ft-kips} \qquad M_{bd} = 3.75 \text{ ft-kips}$$

$$M_{bc} = -22.00 \text{ ft-kips}$$

After the numerical values of the end couples are known, the end shears for each member can be calculated and the ordinary shear and

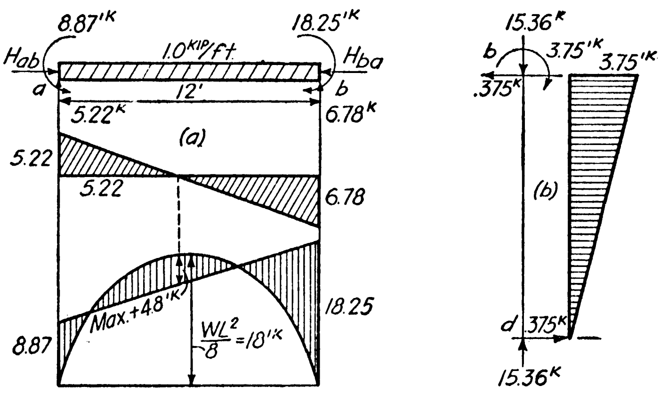


Fig. 37.

bending-moment diagrams drawn. In drawing the bending-moment diagrams, it is essential in design work to use the ordinary conventional signs for the bending moments, which are in terms of the curvature of the member. As the analysis by means of the slope-deflection equations gives the direction of the end couples acting on each member, the direction of the curvature of the axis of the member is known and consequently the usual bending-moment diagram is easily drawn. For example, from the frame of Fig. 36 we would obtain the diagrams in Fig. 37.

The axial forces acting on members *ab* and *bc* cannot be determined unless the horizontal displacement of the joint *b* is considered, and, as this movement would affect the values of the end moments, the above solution is not exact. However, it is sufficiently correct for all practical purposes.

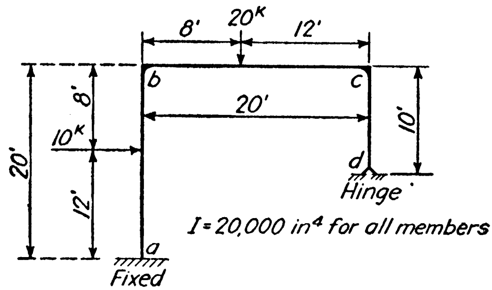


FIG. 38.

Example 11. The frame shown in Fig. 38 represents a structure which has neither angular nor linear displacement at support *a* but is free to rotate at *d*. The members are rigidly connected at joints *b* and *c*, which have both rotation and horizontal displacement. Vertical motion of joints *b* and *c* due to change in length of the members will be neglected. The end moments can be expressed in terms of the displacement of the joints by means of equations 25 and 26. It should be noted that the length is taken in foot units and therefore the values of θ and Δ that are obtained are actually $\dfrac{E\theta}{12}$ and $\dfrac{E\Delta}{144}$.

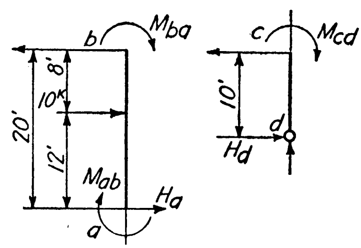


FIG. 39.

These relative values are more convenient than the absolute values when only the end moments are desired.

$$M_{ab} = (2)\left(\frac{20{,}000}{20}\right)\theta_b - (6)\left(\frac{20{,}000}{20}\right)\left(\frac{\Delta}{20}\right) - \frac{(10)(8)(12)(8)}{(20)(20)}$$

or

$$
\begin{aligned}
M_{ab} &= 2000\theta_b - 300\Delta - 19.2 \\
M_{ba} &= 4000\theta_b - 300\Delta + 28.8 \\
M_{bc} &= 4000\theta_b + 2000\theta_c - 57.6 \\
M_{cb} &= 2000\theta_b + 4000\theta_c + 38.4
\end{aligned}
\qquad (34)
$$

$$M_{cd} = 3\left(\frac{20{,}000}{10}\right)\theta_c - (3)\left(\frac{20{,}000}{10}\right)\left(\frac{\Delta}{10}\right)$$

or

$$M_{cd} = 6000\theta_c - 600\Delta$$

The displacement Δ is assumed to the right and will therefore produce negative moments. However, it could just as well have been assumed to the left, that is, $+6K\dfrac{\Delta}{L}$ could have been used. Then Δ would have come out negative.

The end moments must satisfy the following equilibrium equations:

$$M_{ba} + M_{bc} = 0 \tag{35a}$$

$$M_{cb} + M_{cd} = 0 \tag{35b}$$

$$H_a + H_d + 10 = 0 \tag{35c}$$

Equations 35a and 35b are expressed directly in terms of the end moments so that only equation 35c need be modified. By considering the equilibrium of members ab and cd for the forces shown in Fig. 39 we can write

$$H_a = \frac{M_{ab} + M_{ba} - 80}{20}$$

$$H_d = \frac{M_{cd}}{10}$$

By substituting the values of H_a and H_d in equation 35c this equation is then expressed in terms of the end moments.

$$\frac{M_{ab} + M_{ba} - 80}{20} + \frac{M_{cd}}{10} + 10 = 0 \tag{35c'}$$

or

$$M_{ab} + M_{ba} + 2M_{cd} = -120 \tag{35d}$$

When the values of the moments in equations 35a, b, and d are expressed in terms of the displacements of the joints by means of equations 34, the following equations are obtained:

$$\begin{aligned} 8000\theta_b + 2000\theta_c - 300\Delta &= 28.8 \\ 2000\theta_b + 10{,}000\theta_c - 600\Delta &= -38.4 \\ 6000\theta_b + 12{,}000\theta_c - 1800\Delta &= -129.6 \end{aligned} \tag{36}$$

Solving these equations, we obtain

$$\theta_b = 0.0072$$

$$\theta_c = 0.0008$$

$$\Delta = 0.1013$$

Substituting these values in equations 34 gives

$$M_{ab} = -35.2 \text{ ft-kips} \qquad M_{cb} = 56.0 \text{ ft-kips}$$

$$M_{ba} = 27.2 \text{ ft-kips} \qquad M_{cd} = -56.0 \text{ ft-kips}$$

$$M_{bc} = -27.2 \text{ ft-kips}$$

Example 12. The frame in Fig. 40 will illustrate a problem that involves a cantilever member and an eccentric load on a column, two conditions that are often encountered. The calculation of the fixed-end moments for the eccentric load will be explained although the

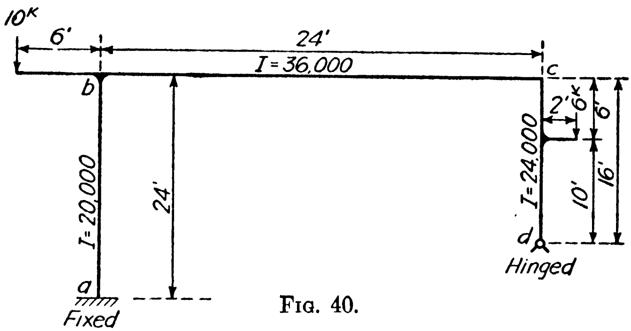


FIG. 40.

method has already been discussed in Article 15. An arrangement of the numerical solution that is particularly convenient is to determine first the angular rotation of the end tangents due to the applied loads on the member for the conditions of free or hinge supports and then by means of equations 29a and b to compute those end moments that would produce equal but opposite angles. These end moments are the fixed-end moments, as they will prevent the tangents from rotating when the loads are applied.

For determining α_a and α_b, the conjugate-beam method is most convenient. As this method is explained in Chapter II, it will be applied here without further description.

If member cd is considered simply supported at both c and d, the bending-moment diagram is given by Fig. 41a.

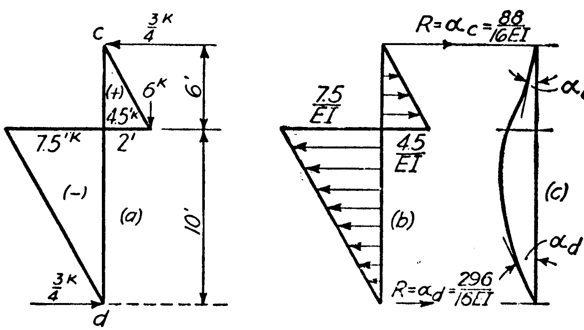


FIG. 41.

If this $\dfrac{M}{EI}$ diagram is considered as an applied load (Fig. 41b) on the conjugate beam, the reactions of this beam are numerically equal to

the rotation of the end tangents of the original beam, or

$$\alpha_c = \frac{88}{16EI} \quad \text{(clockwise)} \qquad \alpha_d = -\frac{296}{16EI} \quad \text{(counterclockwise)}$$

For the beam fixed at c and hinged at d

$$M_{Fcd} = 3\left(\frac{EI}{L}\right)\left(-\frac{88}{16EI}\right) = -\frac{(3)(88)}{(16)(16)} = -1.03 \text{ ft-kips}$$

if both c and d are fixed; then, by equation 29a,

$$M_{Fcd} = 4\left(\frac{EI}{16}\right)\left(-\frac{88}{16EI}\right) + 2\left(\frac{EI}{16}\right)\left(\frac{296}{16EI}\right) = +0.935 \text{ ft-kip}$$

The slope-deflection equations for all end moments in Fig. 40 can now be written as follows:

$$M_{ab} = 2\left(\frac{20{,}000}{24}\right)\theta_b + 6\left(\frac{20{,}000}{24}\right)\left(\frac{\Delta}{24}\right) = 1667\theta_b + 208.3\Delta$$

$$M_{ba} = 3334\theta_b + 208.3\Delta$$

$$M_{bc} = \left(\frac{36{,}000}{24}\right)(4\theta_b + 2\theta_c) = 6000\theta_b + 3000\theta_c \tag{37}$$

$$M_{cb} = 3000\theta_b + 6000\theta_c$$

$$M_{cd} = \left(\frac{24{,}000}{16}\right)\left(3\theta_c + \frac{3\Delta}{16}\right) - 1.03 = 4500\theta_c + 281.3\Delta - 1.03$$

The end moments must satisfy the following equilibrium conditions:

$$M_{ba} + M_{bc} + 60 = 0 \tag{38a}$$

$$M_{cb} + M_{cd} = 0 \tag{38b}$$

$$H_a + H_d = 0 \tag{38c}$$

Equation 38c can be written

$$\frac{M_{ab} + M_{ba}}{24} + \frac{M_{cd} + 12}{16} = 0 \tag{38d}$$

Substituting the value of the moments from equation 37 in equations 38a, b, and d gives

$$9334\theta_b + 3000\theta_c + 208.3\Delta = -60 \tag{39a'}$$

$$3000\theta_b + 10{,}500\theta_c + 281.3\Delta = 1.03 \tag{39b'}$$

$$5000\theta_b + 6750\theta_c + 838.6\Delta = -16.46 \tag{39c'}$$

A solution of these equations gives

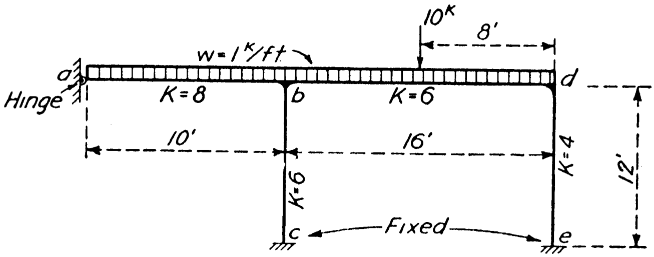
$$\theta_b = -0.00725 \qquad \theta_c = 0.00196 \qquad \Delta = 0.00783$$

and when these values are substituted in equation 37, we obtain

$$M_{ab} = -10.5 \qquad M_{cb} = -10.0$$
$$M_{ba} = -22.6 \qquad M_{cd} = +10.0$$
$$M_{bc} = -37.6$$

PROBLEMS

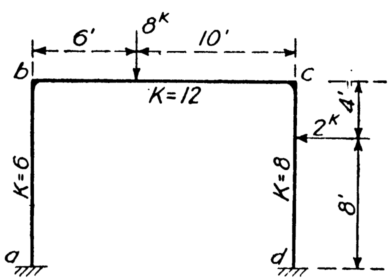
29. (*a*) Compute the end moments for the frame shown by means of the slope-deflection equations.



PROBLEM 29.

(*b*) Draw the shear and bending-moment diagrams for all members.
(*c*) Show the deformed position of the frame by a colored line.
Ans. $M_{ba} = +27.0$ ft-kips; $M_{bd} = -41.4$ ft-kips; $M_{db} = +19.4$ ft-kips.

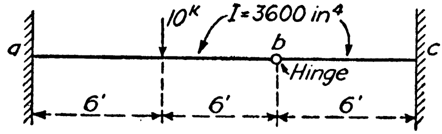
30. Determine the end moments by means of the slope-deflection equations, and show the elastic curve of the structure by a colored line.
Ans. $M_{ba} = +11.04$ ft-kips; $M_{dc} = +3.36$ ft-kips.



PROBLEM 30.

31. If the member *abc* has restrained supports (not fixed) at *a* and *c* and a movable hinge at *b*, express the moments at *a* and *c* in terms of the rotations

θ_a and θ_c only. Keep lengths in foot units. (Suggestion: Write the slope-deflection equations for moments M_{ab} and M_{cb}, and then eliminate Δ_b by satisfying the shear condition at b.) *Ans.* $M_{ab} = 800\theta_a + 400\theta_c - 26.67$.



PROBLEM 31.

Moment-Distribution or Cross Method

22. Moment-Distribution Method. A method of successive approximations that is especially useful in the analysis of continuous frame structures was presented in 1932 by Professor Hardy Cross (*Trans. Am. Soc. C. E.*, Vol. 96). This method, which is commonly referred to as the moment-distribution or Cross method, can be applied directly to continuous structures that are acted upon by force systems that will prevent any *translation* of the joints. In other words, the frame must be supported so that the fixed-end moments in equations 25 and 26 are modified only by the rotations θ of the joints. In calculating the corrections that must be added to the fixed-end moments because of the rotation of the joints, it is possible to carry out the numerical operations by recording the change in the moments produced by the successive rotation of each joint. The entire frame can therefore be analyzed if a convenient method for determining the end moments due to the rotation of *one* joint is known, as that operation can be repeated until all the necessary strain and equilibrium conditions are satisfied.

Let us consider the frame shown in Fig. 42, in which the joint x (any joint) is allowed to rotate until it is in equilibrium while ends a, b, and c are fixed and d is hinged. As no translation during the rotation is permitted, any change in length of the members must be neglected or corrected later.

The moments at the ends of the members meeting at the joint x can be expressed in the usual form by equations 25 and 26.

$$\begin{aligned} M_{xa} &= M_{Fxa} + 4K_1\theta_x \qquad & M_{xb} &= M_{Fxb} + 4K_2\theta_x \\ M_{xc} &= M_{Fxc} + 4K_3\theta_x \qquad & M_{xd} &= M'_{Fxd} + 3K_4\theta_x \end{aligned} \tag{40}$$

in which

$$M'_{Fxd} = M_{Fxd} - \tfrac{1}{2}M_{Fdx}$$

For equilibrium,

$$M_{xa} + M_{xb} + M_{xc} + M_{xd} = 0$$

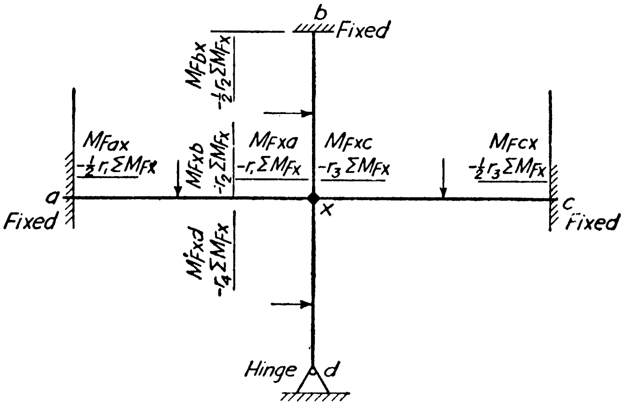


FIG. 42. Arrangement of values for moment-distribution method.

Substituting the value of the moments in the equilibrium equation gives

$$(4K_1 + 4K_2 + 4K_3 + 3K_4)\theta_x + (M_{Fxa} + M_{Fxb} + M_{Fxc} + M'_{Fxd}) = 0$$

Let

$$\Sigma CK = 4K_1 + 4K_2 + 4K_3 + 3K_4$$

where C equals 3 or 4, depending upon whether the opposite end of the member is hinged or fixed.

If

$$\Sigma M_{Fx} = M_{Fxa} + M_{Fxb} + M_{Fxc} + M'_{Fxd}$$

then

$$\theta_x = -\frac{\Sigma M_{Fx}}{\Sigma CK}$$

and the value of the moments will therefore be

$$\begin{aligned} M_{xa} &= M_{Fxa} + \frac{4K_1}{\Sigma CK}(-\Sigma M_{Fx}) = M_{Fxa} + r_1(-\Sigma M_{Fx}) \\ M_{xb} &= M_{Fxb} + \frac{4K_2}{\Sigma CK}(-\Sigma M_{Fx}) = M_{Fxb} + r_2(-\Sigma M_{Fx}) \\ M_{xc} &= M_{Fxc} + \frac{4K_3}{\Sigma CK}(-\Sigma M_{Fx}) = M_{Fxc} + r_3(-\Sigma M_{Fx}) \\ M_{xd} &= M'_{Fxd} + \frac{3K_4}{\Sigma CK}(-\Sigma M_{Fx}) = M'_{Fxd} + r_4(-\Sigma M_{Fx}) \end{aligned} \tag{41}$$

The above equations can be stated in the following way: **the correction that must be added to the fixed-end moment acting on any member at a**

particular joint, because of the rotation of that joint, is equal to the ratio $r = \dfrac{CK}{\Sigma CK}$ times minus the algebraic sum of all the fixed-end moments at the joint. It should be always kept in mind that we are considering the end moments acting on the members and that clockwise moments are taken positive.

If the other end of the member has been held fixed during the rotation of joint x, then the fixed-end moment at that end has also been changed by the rotation θ_x, as expressed by the equation

$$M_{ax} = M_{Fax} + 2K_1\theta_x = M_{Fax} + \tfrac{1}{2}r_1(-\Sigma M_{Fx}) \tag{42}$$

The correction at the end a due to the rotation θ_x is therefore one-half of the correction at the x end (see Fig. 32b). The corrections at b and c are also one-half of the corresponding corrections at x for members bx and cx. No moment is caused at point d as the correction at x for the member xd was calculated for a hinge condition at d. In other words, if a member has a hinged end, no moments need be considered at that end due to the rotation of any joint.

As the various joints can be rotated separately and the moments due to the rotation recorded as shown in Fig. 42, the true value of the end moment is approached as the correction $r(-\Sigma M_{Fx})$ approaches zero. The numerical procedure is most easily explained by specific examples.

Example 13. The frame shown in Figs. 36 and 43 can be solved directly by equations 41 and 42 as only the rotation at joint b affects

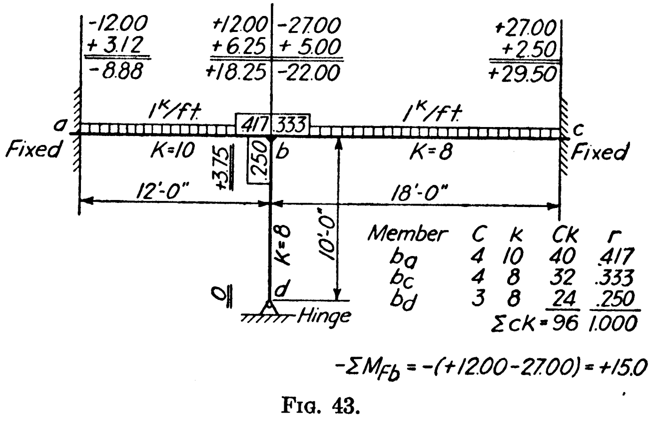


FIG. 43.

the fixed-end moments. Particular attention should be given to the arrangement in Fig. 43 for computing the distribution factors r, as this procedure is especially desirable when members with variable

moments of inertia are used. The algebraic sum of the fixed-end moments at joint b equals $-27.0 + 12.0$ or -15.0. Therefore a moment of r (distribution factor) times $+15.0$ is added to each member at joint b. Thus, the correction for M_{ba} equals $(0.417)(15)$ or 6.25, and the correction added to M_{ab} is half of this value or 3.12. Notice that both corrections have the same sign.

Example 14. When more than one joint must be considered, the above procedure is repeated until the necessary strain and equilibrium conditions are satisfied for all joints. This numerical procedure will be illustrated by computing the end moments for the continuous beam in Fig. 44. The fixed-end moments are determined first, which should not

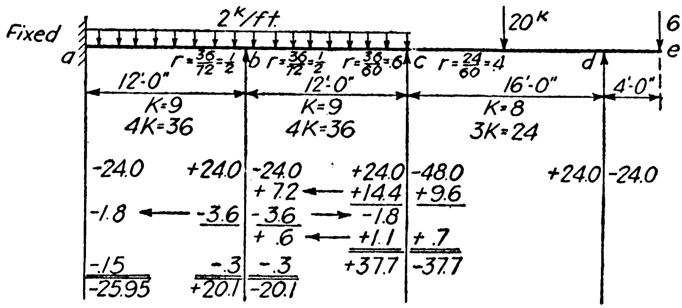


FIG. 44.

cause any difficulty except possibly for M_{Fcd}. This moment is computed on the assumption that there is no resistance to rotation at d when the 20-kip load is applied to the span cd, and therefore the value of M_{Fcd} for that load is $-\frac{3}{16}PL$ or -60 ft-kips. The moment M_{dc} applied to the span cd from the overhanging end is 24 ft-kips, and, therefore, a fixed-end moment of one-half of 24 must also act at c to prevent rotation. The total fixed-end moment M_{Fcd} is therefore the sum of these values, or -48.0 ft-kips.

The distribution factors at b are determined from the CK or $\frac{CI}{L}$ values for members ba and bc. In this case, a value of C equals 4 is used for both, as θ_a and θ_c are kept fixed while joint b rotates to a condition of equilibrium. The distribution factors at c are computed for a value of $4K$ for cb and $3K$ for cd, since θ_b and M_d are kept constant while joint c is rotated. Joint c is balanced first by distributing $-(-48.0 + 24.0)$, which is equal to 24.0 according to the r ratios. The fixed-end moment required at b to keep θ_b constant is one-half of 14.4 or 7.2. This fixed-end moment should be added in with the other fixed-end moments. The unbalanced moment at joint b is now 7.2,

and therefore a -7.2 is distributed according to the r values. The moments required at a and c to keep θ_a and θ_c fixed are $\frac{1}{2}(-3.6)$ or -1.8. Joint c is again unbalanced, and the first operation is repeated. It should be apparent that if this operation is repeated the corrections will steadily decrease and therefore the moments will converge to their true values. Mathematical proof that the series is convergent can be given but seems unnecessary.

Example 15. The moment-distribution method will also be used to analyze the frame shown in Figs. 38 and 45. As only the change in

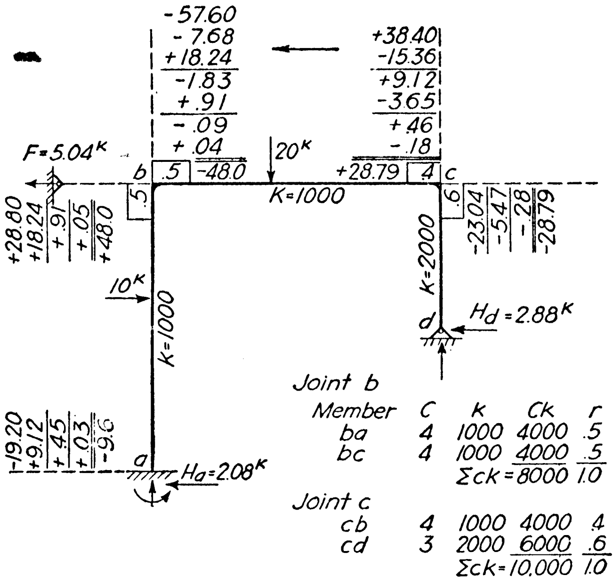


FIG. 45.

end moments due to the rotation of the joints can be considered in the moment-distribution method, an auxiliary force F must be applied, as shown in Fig. 45, to prevent any horizontal movement of b and c. The fixed-end moments due to the loading are then corrected for the successive rotation of the joints b and c. Joint c is balanced first, and then joint b, the various operations being indicated in Fig. 45. After the end moments are determined, the value of the auxiliary force F is calculated from the equilibrium condition

$$H_a + H_d + F - 10 = 0$$

or

$$F = 5.04 \text{ kips}$$

The force F must now be removed from the force system by applying an equal and opposite force.

To determine the moments caused by a force equal but opposite to F, the following procedure will be used. Assume that the joints b and c are moved horizontally to the right a fixed amount by a force F' (see Fig. 46a). If no rotation of the joints is to take place, a fixed-end moment of $6K\left(\frac{E\Delta}{L}\right)$ is required at each end of the member ab and a

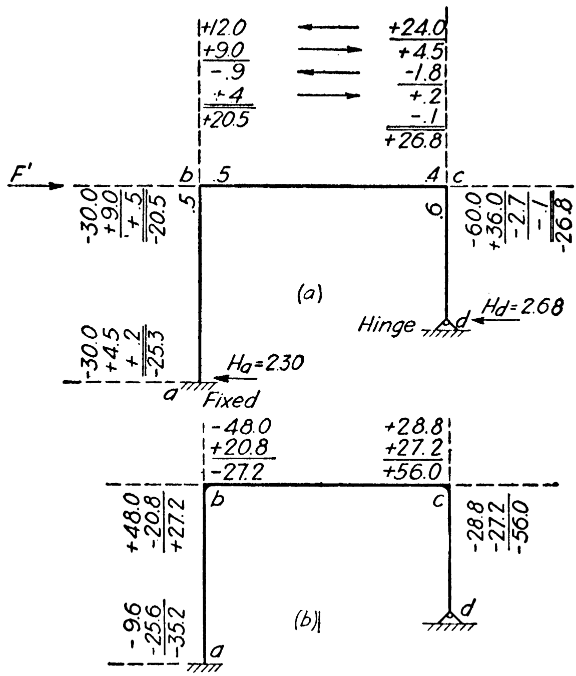


FIG. 46.

moment of $3K\left(\frac{E\Delta}{L}\right)$ at joint c for member cd. These fixed-end moments are corrected for the rotation of the joints in the same manner as were the fixed-end moments for the transverse loading in Fig. 45. Thus, if a displacement of $E\Delta = 0.10$ is taken to the right, the fixed-end moments at both ends of ab are

$$M_{Fab} = M_{Fba} = -\frac{(6)(1000)(0.10)}{20} = -30$$

and at joint c for member cd

$$M_{Fcd} = -\frac{(3)(2000)(0.10)}{10} = -60$$

These fixed-end moments are corrected for rotation of the joints as shown in Fig. 46a. If the actual value of Δ is to be used, it must be kept in the same units as E, I, and L; but, if only the moments that will provide equilibrium are required, any choice of units can be made, provided that such use is consistent throughout the calculations.

The value of the force F' necessary to give the displacement $E\Delta$ equal to 0.10 and the moments recorded in Fig. 46a is equal to

$$F' = H_a + H_d = 2.30 + 2.68 = 4.98$$

However, a force $-F$ equal to 5.04 kips was required, which, by Hooke's Law, can be obtained by increasing the value of F' and the corresponding moments by

$$\frac{F}{F'} = \frac{5.04}{4.98}$$

Consequently, the true value of the moment M_{ab} will be the algebraic sum of -9.6 ft-kips, the moment caused by the actual force system including F, and

$$\frac{5.04}{4.98}(-25.3) = -25.6$$

which is the moment due to $-F$. The final values of the moments are recorded in Fig. 46b.

23. Effect of Shearing Deformation. The deformation due to shearing stresses *reduces* the CK, or stiffness factor, and the carry-over factor. The fixed-end moments are also modified when the loading is unsymmetrical. The amount of the reduction is readily calculated by including the strain energy due to shear in the solution of the problems that are given in Fig. 32a, b, c, and d. Thus in Fig. 32b the total strain energy is

$$U = \int_0^L \frac{(M'_{ab} - V'_a x)^2\,dx}{2EI} + \int_0^L \frac{V'^2_a\,dx}{2AG} \qquad (43a)$$

By applying Castigliano's theorem we obtain

$$\Delta_a = \frac{\partial U}{\partial V'_a} = \int_0^L \frac{(M'_{ab} - V'_a x)(-x)\,dx}{EI} + \int_0^L \frac{V'_a\,dx}{AG} = 0$$

from which (43b)

$$V'_a = \frac{M'_{ab}}{L}\left(\frac{1.5}{1+j}\right)$$

where

$$j = \frac{3EI}{L^2AG}$$

The relationship between the end moment and rotation θ_a of the end section is

$$\theta_a = \frac{\partial U}{\partial M'_{ab}} = \int_0^L \frac{(M'_{ab} - V'_a x)\, dx}{EI} \tag{43c}$$

from which

$$M'_{ab} = \left(\frac{1 + j}{0.25 + j}\right) K\theta_a \qquad \text{and} \qquad M'_{ba} = \left(\frac{0.5 - j}{1 + j}\right) M'_{ab} \tag{43d}$$

If $M'_{ba} = 0$ (hinge at b),

$$M'_{ab} = \frac{3}{1 + j} K\theta_a \tag{43e}$$

From equations 43*d* and 43*e*, the stiffness factors CK are

$$\left(\frac{1 + j}{0.25 + j}\right) K \cdots \text{instead of } 4K \text{ for fixed ends} \tag{43f}$$

$$\left(\frac{3}{1 + j}\right) K \cdots \text{instead of } 3K \text{ for hinged ends} \tag{43g}$$

and the carry-over factor is

$$\frac{0.5 - j}{1 + j} \text{ instead of } 0.5 \tag{43h}$$

For steel beams the ratio $\frac{E}{G}$ is approximately 2.6 and the ratio $\frac{I}{A}$ depends upon the depth of the beam and the shape of the cross section. For I-beams or girders, the area of the web should be used for A which tends to increase the value of j. The effect of shearing deformation is most important for short deep I-sections, but even for them the change in the end moments can usually be neglected.

The fixed-end moments for symmetrical loads on the beam are not changed by shearing deformation. The reason for this is that the end sections remain vertical when the shearing displacements occur as long as the deformation is symmetrical about the center line. Therefore the rotations α of the end cross sections will be due to bending moments only. If $\alpha_a = -\alpha_b$,

$$M_{Fab} = \left(-\frac{1 + j}{0.25 + j}\right) K\alpha_a + \left(\frac{0.5 - j}{0.25 + j}\right) K\alpha_a$$

or

$$M_{Fab} = \left(\frac{-0.5 - 2j}{0.25 + j}\right) K\alpha_a = -2K\alpha_a$$

which is the same as when the shearing force is not considered.

However, when the loads on the beam are unsymmetrical, the rotations α_a and α_b of the end sections may be modified by the shearing deformation. These rotations can be calculated by Castigliano's theorem and the fixed-end moments then calculated by the equations

$$M_{Fab} = \left(\frac{1 + j}{0.25 + j}\right) K(-\alpha_a) + \left(\frac{0.5 - j}{0.25 + j}\right) K(-\alpha_b) \tag{44a}$$

$$M_{Fba} = \left(\frac{0.5 - j}{0.25 + j}\right) K(-\alpha_a) + \left(\frac{1 + j}{0.25 + j}\right) K(-\alpha_b) \tag{44b}$$

The values of α_a and α_b must be used in equations 44*a* and 44*b* with their proper sign.

Example 16. A 24-in. WF @ 74-lb beam is continuous over three spans as shown in Fig. 47. The end moments will be calculated by the

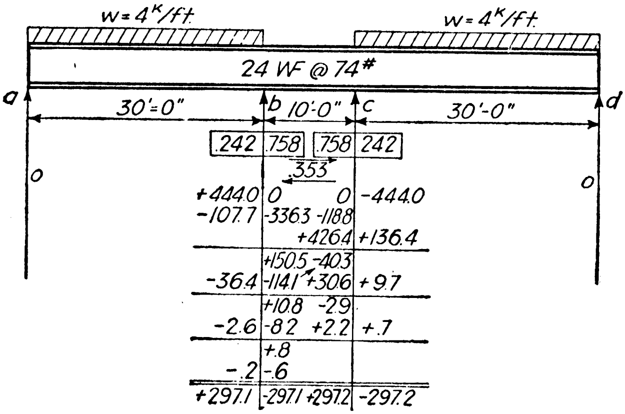


FIG. 47.

moment-distribution method with the effect of shearing deformation included, and these moments will be compared with the ordinary values. The following properties of the I-beam can be obtained from a steel handbook:

$$I = 2034 \text{ in.}^4 \qquad A \text{ (of web)} = 9.7 \text{ in.}^2 \qquad \frac{E}{G} = 2.5$$

$$\text{For span } ab \qquad j = \frac{(3)(2.5)(2034)}{(360)(360)(9.7)} = 0.0121$$

$$\text{For span } bc \qquad j = \frac{(3)(2.5)(2034)}{(120)(120)(9.7)} = 0.109$$

Therefore, the CK values (equations 43f and 43g) for the spans are

$$\text{For } ba \qquad \left(\frac{3}{1+0.012}\right)\left(\frac{2034}{360}\right) = 16.7$$

$$\text{For } bc \qquad \left(\frac{1+0.109}{0.25+0.109}\right)\left(\frac{2034}{120}\right) = 52.3$$

The distribution factors at joints b and c are

$$r_{ba} = \frac{16.7}{16.7+52.3} = 0.242$$

$$r_{bc} = \frac{52.3}{69.0} = 0.758$$

The carry-over factor (equation 43h) for span bc is

$$\frac{0.5-0.109}{1+0.109} = 0.353$$

For a uniform load of 4 kips per foot on span ab the fixed-end moment is

$$M_{Fba} = \left(\frac{3}{1+j}\right)\frac{EI}{L}\alpha_b = \left(\frac{3}{1+0.012}\right)\left(\frac{EI}{L}\right)\frac{wL^3}{(24EI)} = 444 \text{ ft-kips}$$

The distribution of the fixed-end moments as shown in Fig. 47 gives the final moments at b and c as 297.1 ft-kips. If the effect of shear is not included, a value of 300 ft-kips is obtained which shows that the final moments are reduced by only 1 per cent even though the distribution and carry-over factors are reduced as much as 30 per cent. If a concentrated load of 80 kips is placed at the center of span bc, the moments at b and c are 33.3 when shear is neglected and 33.1 when it is considered. Therefore, it appears that in this problem the shearing deformation has practically no effect upon the end moments even though it affects the stiffness and carry-over factors considerably.

Example 17. A plate girder is continuous over three spans as shown in Fig. 48. The moment of inertia I is 20,000 in.^{4}, and the area of the web is 16 in.^{2} The maximum and minimum moments in the girder that are obtained by first considering the shearing deformation and then by neglecting it will be compared.

The values of j and CK for the end and center spans are:

For end spans

$$j = \frac{(3)(2.5)(20{,}000)}{(120)^2(16)} = 0.65$$

For center span

$$j = 0.026 \quad \text{and} \quad CK = \left(\frac{1 + 0.026}{0.25 + 0.026}\right)\left(\frac{20{,}000}{600}\right) = 123.8$$

For end spans

$$CK = \left(\frac{3}{1 + 0.65}\right)\left(\frac{20{,}000}{120}\right) = 303$$

Carry-over for center span

$$\frac{0.5 - 0.026}{1 + 0.026} = 0.462$$

The distribution factors at b and c are

$$r_{ba} = \frac{303}{426.8} = 0.71 \qquad r_{bc} = \frac{123.8}{426.8} = 0.29$$

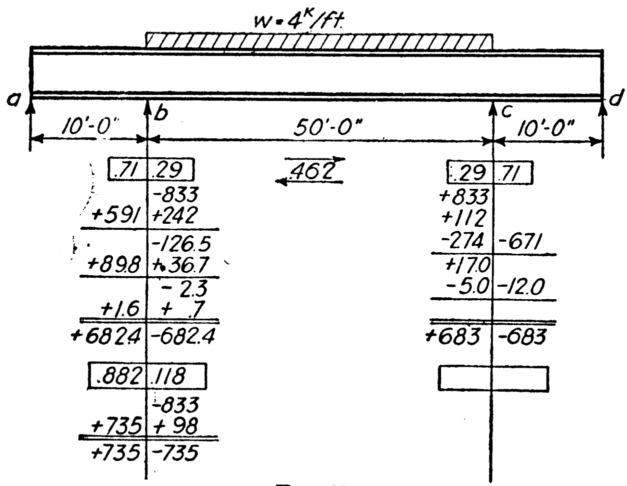


FIG. 48.

Because of symmetry the fixed-end moments are not affected by the shearing deformation and therefore are equal to

$$\frac{wL^2}{12} = \frac{(4)(50)(50)}{12} = \pm 833$$

The numerical operations shown in Fig. 48 follow the usual procedure and give the final bending moments at b and c equal to -683 ft-kips. These end moments give a positive bending moment of 567 ft-kips at the center of span bc.

If the moments are now calculated by the usual distribution and carry-over factors in which the shearing deformation is neglected, the bending moments at b and c are equal to -735 and the positive moment at the center of span bc is 515 ft-kips. Therefore, the shearing deformation decreases the negative bending moments at b and c by 7 per cent and increases the positive moment at the center of span bc by 10 per cent. The distribution factors 0.882 and 0.118 are obtained by taking advantage of the symmetry of the structure. Since $\theta_c = -\theta_b$,

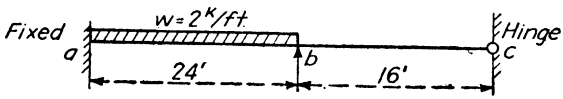
$$M_{bc} = M_{Fbc} + 4K\theta_b + 2K\theta_c = M_{Fbc} + 2K\theta_b$$

Therefore, if C equal to 2 is used for member bc, the problem is solved with one distribution.

PROBLEMS

32. Compute the end moments for the continuous beam abc by the moment-distribution method. Use $I = 240$ in.^{4} for both spans.

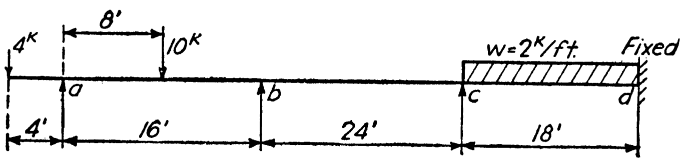
Ans. $M_{ab} = -118.6$ ft-kips.



PROBLEM 32.

33. Determine the end moments for the continuous beam $abcd$ by the moment-distribution method. Use $I = 960$ in.^{4} for all spans.

Ans. $M_{ba} = +3.3$ ft-kips; $M_{cd} = -18.5$ ft-kips.

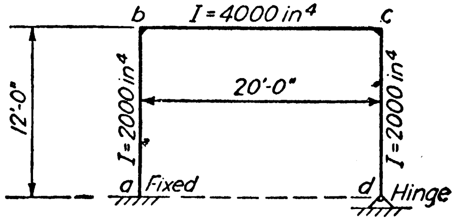


PROBLEM 33.

34. Compute the moments in the frame of Problem 29 by the moment-distribution method.

35. Solve Problem 30 by the moment-distribution method.

36. If the member bc shortens 0.07 in. as the result of a change in temperature, what is the value of the end moments in all members? Use $E = 30 \times 10^6$ lb per in.^{2} *Ans.* $M_{ba} = -5.6$ ft-kips; $M_{cd} = 20.0$ ft-kips.



PROBLEM 36.

(Suggestion: Hold joint b without translation by an auxiliary force F, and then remove F as in Example 15.)

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CHAPTER IV

BUILDING FRAMES SUBJECTED TO VERTICAL LOADS

24. Introduction. At the present time the design of reinforced-concrete buildings and many types of steel structures is based primarily upon a recognition of continuity among the various structural elements. The degree of continuity that can be expected in reinforced-concrete buildings requires careful study, but, in general, when the floors, girders, and columns have been properly reinforced and constructed with adequate supervision, the assumptions already made for rigid frame structures will apply. As continuity can exist in all directions, a building is a *space* structure and must be so regarded if its true structural action is to be properly interpreted. This means that the floors act as slabs and, with the girders, are subjected to bending, shear, and torsion, and that the columns are subjected to direct stress, together with shear and bending in two directions. If no girders are used, but the floor slabs are supported directly on the columns, the building falls in the classification of a flat slab structure which is usually designed by standard coefficients that have been determined from a mathematical and experimental study of slabs under various loading and boundary conditions. The analysis of such structures as equivalent frames has been recommended by some engineers, but as yet this procedure has not been generally adopted.

When the floor slab is supported by beams and girders that frame into columns, the usual procedure is to neglect the torsion in the girders and to calculate the stresses due to the bending in each direction as a two-dimensional problem. Under these conditions the frame becomes similar to the idealized structures that have already been discussed except that the number of members is greatly increased. At this time it should be emphasized that no indeterminate structure is actually designed as a single unit, but, instead, a number of substitute structures, that represent the action of the actual structure as closely as possible, are proportioned and analyzed. The analysis of the substitute structure will be considered first and then its relation to the actual conditions will be discussed.

In buildings with steel framework, the amount and effect of the deformation that may occur in the connections of the beam and girders

to the columns constitute a most important and difficult problem. Most riveted connections will be of a semi-rigid type, although some types can be made to undergo very small deformations under working conditions. Many kinds of welded connections and some riveted connections are used that will give practically no local deformation and therefore come within the assumptions that have been made in the preceding chapters. The following analysis is based upon the assumption of rigid connections, that is, that the end tangents of all members meeting at a joint rotate through the same angle. The problem of semi-rigid connections can be treated better after members with variable moments of inertia have been considered.

25. Selection of the Primary Frames. The frame shown in Fig. 49 is typical of those encountered in many buildings in which torsional

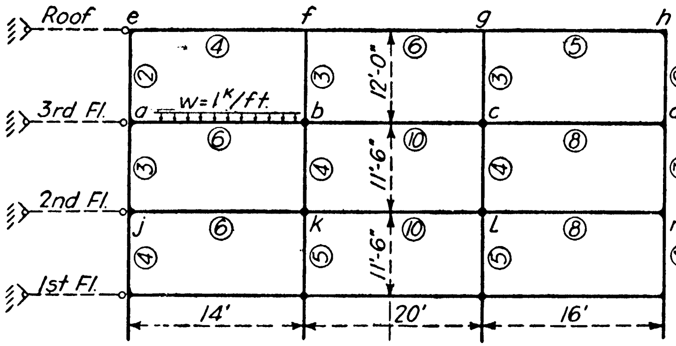

Fig. 49.

action is neglected and in which the connections undergo practically no local deformation. In such frames the vertical loads on the girders will be carried to the foundations largely by the adjacent columns. The end moments acting on any girder are affected by the amount of end restraint, which is primarily a function of the stiffness of the girder as compared to the stiffness of the adjacent members. Horizontal movement of the floors may be of importance in some structures, but for the present it will be assumed that the frame is supported against any sidesway, and, consequently, the fixed-end moments need be corrected only for the rotation of the joints. The removal of the restraining forces, if necessary, will be treated in the following chapter.

When a span of the frame in Fig. 49, such as *ab*, is loaded, the end moments in the various members, due to this load only, will be affected by the rotation of joints *a* and *b* and, to a much less degree, by the rotation of other joints such as *e*, *f*, *c*, *j*, and *k*. Frequently the rotation of these other joints can be neglected or, in other words, the members can

be considered as fixed at those points. A better approximation is often obtained by assuming that these joints do undergo some rotation, the amount of this rotation and its effect upon the moments of the various members being estimated by the conditions stated in the following derivation.

Let any member ab, Fig. 50, be subjected to a moment M_{ab} at the end a, and let the end b be restrained by the members bc, bd, be, that are either fixed as at c and d or hinged as at e. Usually a fixed-end condition should be assumed.

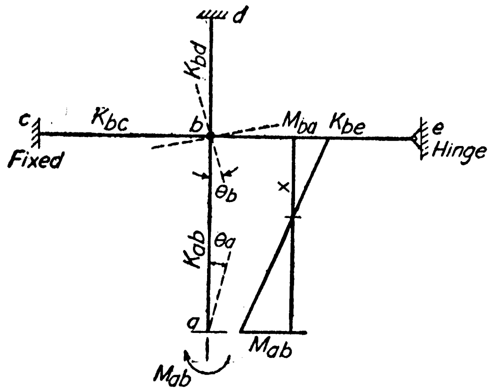

FIG. 50.

Applying the slope-deflection equations to the frame in Fig. 50 gives

$$M_{ab} = K_{ab}(4\theta_a + 2\theta_b) \qquad M_{ba} = K_{ab}(2\theta_a + 4\theta_b)$$

$$M_{bc} = 4K_{bc}\theta_b \qquad M_{bd} = 4K_{bd}\theta_b \qquad M_{be} = 3K_{be}\theta_b$$

Since $\Sigma M_b = 0$,

$$\theta_b(4K_{ab} + 4K_{bc} + 4K_{bd} + 3K_{be}) + 2K_{ab}\theta_a = 0$$

or, using the notation of Article 22,

$$\theta_b = -\frac{2K_{ab}}{\Sigma_b CK}\theta_a = -\frac{1}{2}r_{ba}\theta_a$$

where r_{ba} is the distribution factor for the member ab at the joint b. If this value of θ_b is now substituted in the above expressions for M_{ab} and M_{ba}, the following equations will be obtained:

$$M_{ab} = K_{ab}(4\theta_a - r_{ba}\theta_a) = (4 - r_{ba})K_{ab}\theta_a \qquad (45a)$$

$$M_{ba} = K_{ab}(2\theta_a - 2r_{ba}\theta_a) = 2(1 - r_{ba})K_{ab}\theta_a \qquad (45b)$$

Another convenient way of expressing M_{ba} is

$$M_{ba} = \frac{2(1 - r_{ba})}{4 - r_{ba}} M_{ab} \tag{45c}$$

in which

$$\frac{2(1 - r_{ba})}{4 - r_{ba}}$$

is the carry-over factor from a to b. From equations 45a and 45c, the distribution factor at the end a and the carry-over factor from a to b can be obtained for any degree of restraint $(1 - r)$ at the end b. Thus, when the end b is considered fixed, r equals 0, and when hinged, r equals 1. The stiffness factor for the member ab at the end a can therefore vary from $3K$ to $4K$, and the carry-over factor from 0 to $\frac{1}{2}$.

In the analysis of a building frame that is subjected to vertical loads, it is frequently convenient to analyze the structure for the loads on each span separately and then to combine the loadings so as to obtain the maximum and minimum moments. When only one span is loaded, such as ab, Fig. 49, it is often assumed that all joints except a and b remain fixed. In many problems, however, it is better to make use of equations 45a and 45c, which enable a more accurate analysis of the structure to be made. The part of the structure considered in the analysis will be designated the primary frame. It may sometimes be necessary to include more of the structure than is used in this illustration. Equations 45a and 45c are not easily applied to members that have transverse loading.

26. Analysis of the Primary Frame. The end moments acting on the members of the frame in Fig. 51 will be calculated by the moment-distribution method, using the modified stiffness factor $(4 - r)K$ for all

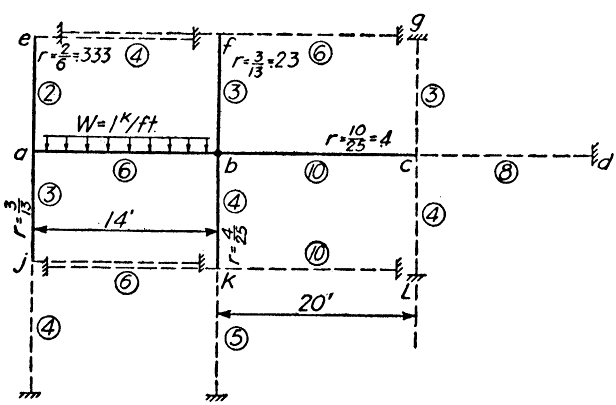

FIG. 51.

members, except *ab*. Although this procedure gives only a slight change in the distribution factor, it makes a considerable difference in the carry-over factors which affect the moments at the joints away from the loaded span. The distribution and carry-over factors for members at joints *a* and *b* when Equations 45*a* and 45*c* are used are:

JOINT *a*

Member	C	K	CK	r	Carry-over
ae	(4–0.333)	2	7.33	0.172	0.362
ab	4	6	24.0	0.563	0.5
aj	(4–0.23)	3	11.31	0.265	0.408
		ΣCK =	42.64	1.000	

JOINT *b*

Member	C	K	CK	r	Carry-over
ba	4	6	24.0	0.277	0.5
bf	(4–0.23)	3	11.31	0.131	0.408
bc	(4–0.4)	10	36.0	0.415	0.333
bk	(4–0.16)	4	15.36	0.177	0.437
		ΣCK =	86.67	1.000	

The numerical solution for the end moments by the moment-distribution method is given in Fig. 52. The fixed-end moment at joint *a*, −16.33 ft-kips, is first distributed, and one-half of the correction for M_{ab} is carried over to M_{Fba}; that is,

$$(\tfrac{1}{2})(9.20) = 4.6$$

is added to the fixed-end moment 16.33 at *b*.

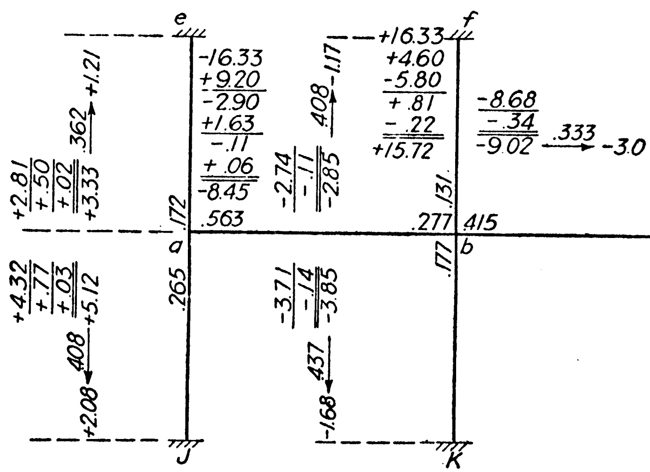

FIG. 52.

The unbalanced moment of 20.93 at b is now distributed, and the correction -5.80 applied to M_{ba} produces a moment of -2.90 at a. Joint a is again balanced, and the procedure is continued until the desired accuracy is obtained.

After the moments at the joints a and b are determined the other end moments can be obtained from the carry-over factors that have already been computed. Thus the moment at e will equal (0.362)(3.33) or 1.21, which is correct within the limits of the conditions represented by Fig. 51.

The moments caused by a uniform load of 1 kip per foot on the span bc are shown in Fig. 53. The solution of this problem is carried out like the preceding one; that is, the distribution factors for the members

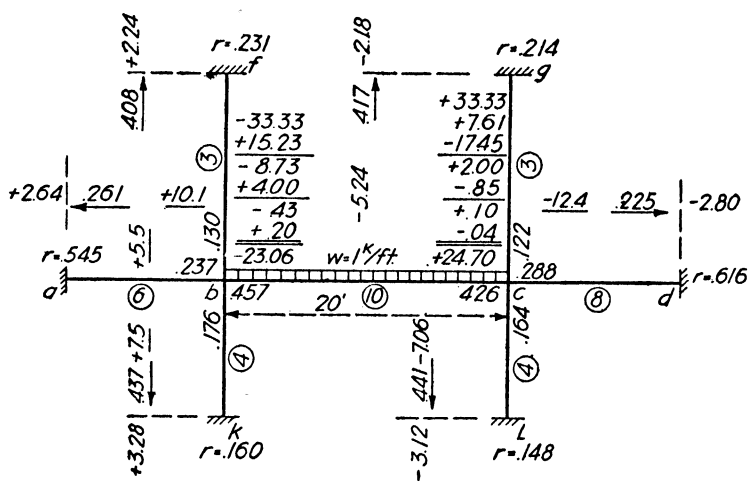

FIG. 53.

framing into joint b and c are calculated for a modified restraint for all members except bc. The carry-over factors have also been determined on the basis of a degree of flexibility at the far ends equal to r.

The moments at the ends of the members that provide the restraining action for the unloaded spans can also be obtained with sufficient accuracy from the above analysis. Thus, in Fig. 52, the moment M_{cb} (-3.0) is resisted by a $+3.0$ in the members cg, cd, and cl, Fig. 51, which is distributed to the three members in proportion to their K values. The value of M_{cd} will therefore be $(\frac{8}{15})(3.0)$ or 1.6 and M_{dc} will be (0.225)(1.6) or 0.36. This use of modified restraints in determining the distribution of moments can be extended further, but in general an extension is not of practical importance.

27. Maximum and Minimum Moments. The preceding analysis mdicates that, although a vertical load on one span of a building frame will produce moments over several bays and in the adjacent stories, in most practical problems only the moments in the adjacent beams and columns need be considered. From the moments that have been computed for a unit load on each span, the combined effect of the dead load

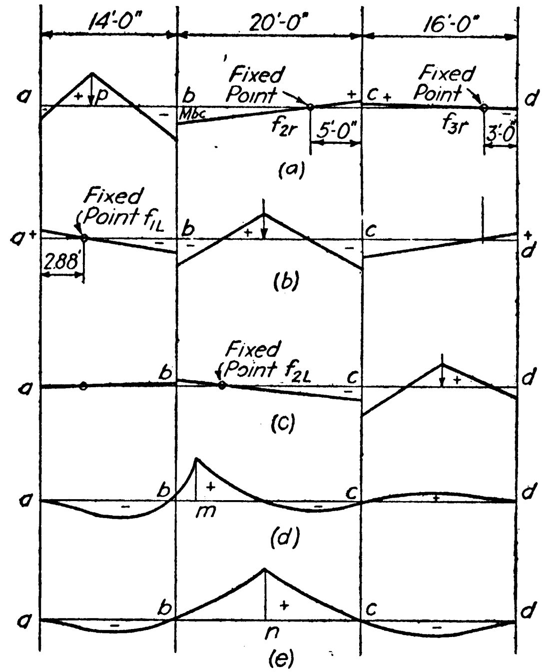

FIG. 54.

of the structure and of the live load (for full span loadings) can be obtained by superposition. However, although only the end moments have thus far been computed, it is necessary that the designer also investigate the moments that may occur at any intermediate section. Such moments can be pictured graphically both by drawing moment diagrams for loads on various spans and by constructing influence diagrams for the moments at various sections of the girders.

In Fig. 54*a*, the moment diagram is shown for a load P on the span *ab*, and in Fig. 54*b* for a load on span *bc*. A study of these moment diagrams yields some interesting information; for instance, when the load P is applied anywhere on span *ab*, the moment diagrams for spans *bc* and *cd* pass through the points f_{2r} and f_{3r}, respectively. These two

points, which are commonly called "fixed points," are sometimes used in the analysis of continuous frames. The position of the fixed points can be obtained directly from equation 45c, since

$$\frac{M_{cb}}{M_{bc} + M_{cb}} = \frac{x}{L}$$

and

$$M_{cb} = \frac{2(1 - r)}{4 - r} M_{bc}$$

From these expressions the distance x of the fixed point from any end whose degree of flexibility is r will be given by the expression

$$x = \frac{2(1 - r)}{3(2 - r)} L \tag{46}$$

For all practical purposes, the value of r can be taken equal to the ordinary distribution factor.

Each span will have two fixed points that are useful in determining the nature of the moments that are produced in that span by loads on adjacent spans. For convenience, any span such as bc can be considered in three parts: the portion between the fixed points f_{2l} and f_{2r}, which is subjected only to negative moments from either adjacent span; the portion between the fixed point f_{2r} and the support c, which is subjected to positive moments from loads on span ab and negative moments from loads on span cd; and the portion between f_{2l} and the support b, which is subjected to negative moments from loads on span ab and positive moments from loads on span cd.

The most important features of the diagrams in Fig. 54 can be summarized as follows:

(*a*) Influence diagrams for moment for sections between a fixed point and the support will be of the type shown in Fig. 54*d* for point m.

(*b*) Influence diagrams for sections between the fixed points will be of the type shown in Fig. 54*e* for point n.

(*c*) For points such as m, the maximum and minimum moments will be obtained for partial loading of the span itself and with one adjacent span loaded.

(*d*) For point n, the maximum moments will occur when the span itself is fully loaded together with alternate spans. The moment caused by the loads on the alternate spans can usually be neglected as it is comparatively small. The minimum moment at point n will occur with no live load on the span bc but with the adjacent spans ab and cd fully loaded.

(*e*) If each span is loaded separately with a unit uniform load as in the preceding analysis, the separate values for dead and live loads can be combined to give actual maximum and minimum values for all sections between the fixed points and at the supports, but only approximate values for sections between the supports and the fixed points.

Example 18. The maximum and minimum moment curves will be drawn for the spans *ab* and *bc* of the frame in Fig. 49 for a uniform dead load of 2 kips per foot and a uniform live load of 4 kips per foot. The end moments for a unit uniform load on spans *ab* and *bc* have already been calculated in Figs. 52 and 53. Therefore, the moments at intermediate points can be computed from the conditions for statical equilibrium, or by scaling the ordinates to the bending-moment diagrams. The effects of loads on the floors above and below will be neglected, although they can readily be included and for the exterior spans it is sometimes necessary to do so. In Table 2 the ordinates y_1, y_2, y_3 give the values of the moments at various sections in span *bc* for a uniform load of 1 kip per foot on spans *ab*, *bc*, and *cd*, respectively. The dead-load moment is given by column 5, and the maximum and minimum live-load moments by columns 6 and 7. The maximum and minimum combined values are recorded in columns 8 and 9.

The above values are shown graphically by the curves in Fig. 55. The solid portions of the curves between the fixed points represent actual maximum and minimum values; the dotted portions are estimated. The error involved in the dotted portion, however, is relatively small.

TABLE 2

POINT	y_1	y_2	y_3	$2\Sigma y$	$4\Sigma + y$	$4\Sigma - y$	Max	Min
1	2	3	4	5	6	7	8	9
$6r$	−9.02	−23.06	3.52	−57.12	14.08	−128.32	−43.04	−185.44
7	−7.0	4.50	1.08	−2.84	22.32	−28.00	19.48	−30.84
f_{2l}	−6.12	13.00	0	13.76	52.0	−24.48	65.8	−10.72
8	−5.0	20.91	−1.38	29.06	83.64	−25.52	112.70	3.54
9	−3.0	26.12	−3.83	38.58	104.48	−27.32	143.06	11.26
10	−1.0	20.33	−6.29	26.08	81.32	−29.08	107.40	−3.00
f_{2r}	0	13.0	−7.50	11.0	52.0	−30.0	63.0	−19.0
11	1.0	3.34	−8.74	−8.80	17.36	−34.96	8.56	−43.76
12	3.0	−24.75	−11.20	−65.90	12.00	−143.80	−53.90	−209.70

From a study of the curves for maximum and minimum moments, Fig. 55, it can be seen that large variations in bending moments are obtained at the various sections and that the position of the points of contraflexure vary over a considerable distance. The amount of this

variation depends upon the arrangement of the spans and upon the ratio of the live to the dead load.

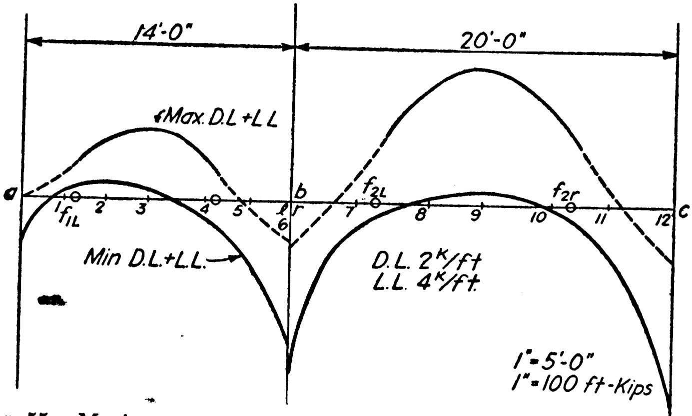

FIG. 55. Maximum and minimum bending moments in a building frame.

28. Maximum Shear. The maximum shear that can occur at any section of a girder in a continuous building frame will be largely due to the loads on that span except for unusual span arrangements. Nevertheless the effect of continuity upon the shear should be studied, at least to the extent of knowing the error involved if the loads on adjacent spans are neglected. The influence diagram for the shear at any section m of an interior girder in a continuous frame, as shown in Fig. 56, is useful in studying the problem. The shape of the influence diagram can be easily sketched by the use of the reciprocal theorem as illustrated in Chapter II. The problem of determining the maximum shear is studied best by considering, first, the shear due to loads on the span itself, and second, the shear from loads on one of the adjacent spans.

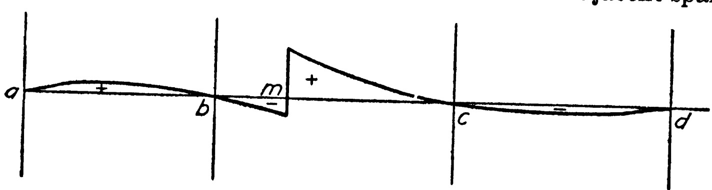

FIG. 56.

The influence diagram in Fig. 56 shows that the maximum positive live-load shear for a uniform load on span bc occurs when the load is placed on part of the beam. This condition is the same as for a simply supported beam. The effect of end restraint upon the maximum shear is

shown in Fig. 57*a*, in which maximum shear curves have been drawn for uniform loading on the span *bc* for three different end conditions, namely, simply supported at both ends, fixed at both ends, and simply supported at one end and fixed at the other. These curves were drawn for a live load equal to twice the dead load. A study of these maximum shear curves shows that the difference in end restraints and not the degree of restraint is the more important factor. The values for both ends simply supported differ but slightly from those for both ends fixed, whereas when one end is simply supported and the other fixed there is a noticeable difference. This comparison indicates that, for interior

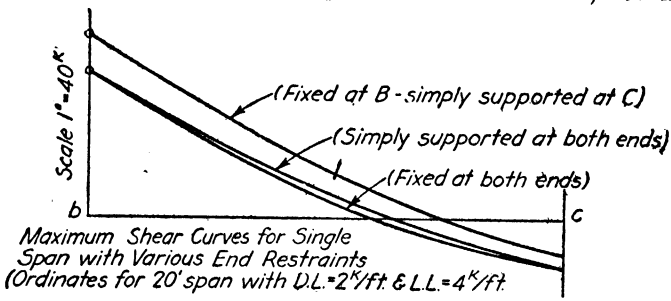

FIG. 57*a*.

spans, the shear from loads on the span itself is practically the same as for a simply supported beam, whereas for exterior spans some allowance must be made for the increased shear at the interior support. An increase of 20 per cent over the maximum shear for a simply supported span will usually be a sufficient allowance for the end shear, but at intermediate sections a larger allowance should be made.

The shear due to loads on adjacent spans can be calculated directly from the end moments that have been determined. This effect is relatively small unless the adjacent spans are large as compared to the span itself. A more important action is the fluctuation in the shear values that takes place if any unequal settlement of the supports occurs, and consequently some reserve strength to allow for such redistribution of shear is desirable.

29. Maximum Stress in Columns. As the end shear in the girders must be resisted by the axial stress in the columns, the latter can be determined directly from the shear values. The maximum axial load in any column will be obtained when the adjacent bays are fully loaded in all stories. For exterior columns such loading will also give large bending moments whereas for interior columns the maximum bending moments will be obtained with a different loading arrangement. In other words, maximum direct stress and maximum bending moment do not

occur simultaneously for interior columns. The most critical stress conditions will vary for different span arrangements.

The maximum moment in an exterior column such as *aj* (Fig. 49) can be calculated with sufficient accuracy by loading spans *ab* and *jk;* for the interior columns, such as *bk,* by loading *bc* and *kl.* The numerical values can be obtained by combining the moments that have already been computed for the loads on each span.

Example 19. The maximum value of V_{ab}, V_{ba}, and V_{bc} for the building frame in Fig. 49 will be computed for a dead load of 2 kips per foot and a live load of 4 kips per foot, and maximum shear curves will be drawn.

The maximum value of V_{ab} occurs with both dead and live load on spans *ab* and *cd* but with only dead load on *bc*. By using the end moments that are recorded in Figs. 52 and 53 for spans *ab* and *bc* and by making similar calculations for a uniform load on *cd*, the following values are obtained:

$$\begin{aligned}
&\text{From span } ab \qquad V_{ab} = 42.0 - \frac{(6)(15.72 - 8.45)}{14} = 38.9\\
&\text{From span } bc \qquad V_{ab} = -\frac{(10.1 + 2.64)(2)}{14} = -1.8\\
&\text{From span } cd \qquad V_{ab} = \frac{(6)(1.63 + 0.42)}{14} = 0.9\\
&\text{Total } V_{ab} = 38.0
\end{aligned}$$

For maximum value of V_{ba}, full dead and live loads are placed on spans *ab* and *bc* while dead load only is placed on span *cd*.

$$\begin{aligned}
&\text{From span } ab \qquad V_{ba} = -42 - \frac{(6)(15.72 - 8.45)}{14} = -45.1\\
&\text{From span } bc \qquad V_{ba} = -\frac{(6)(10.1 + 2.64)}{14} = -5.5\\
&\text{From span } cd \qquad V_{ba} = +\frac{(2)(1.63 + 0.42)}{14} = 0.3\\
&\text{Total } V_{ba} = -50.3
\end{aligned}$$

For maximum value of V_{bc}, dead and live loads are placed on spans *ab* and *bc* while only dead load is on *cd*.

$$\begin{aligned}
&\text{From span } ab \qquad V_{bc} = \frac{(6)(9.02 + 3.0)}{20} = 3.6\\
&\text{From span } bc \qquad V_{bc} = 60 - \frac{(6)(24.7 - 23.1)}{20} = 59.5\\
&\text{From span } cd \qquad V_{bc} = -\frac{(2)(11.2 + 3.52)}{20} = -1.5\\
&\text{Total } V_{bc} = 61.6
\end{aligned}$$

The above values of V_{ab}, V_{ba}, and V_{bc} vary by -9.5 per cent, $+19.8$ per cent and $+2.7$ per cent, respectively, from the corresponding values that would be obtained for simply supported beams.

The maximum positive and negative shear at the center of span *ab* for the live load on span *ab* can be determined with sufficient accuracy by taking the maximum positive live load shear as one-fourth of the live load end shear V_{ab}, and the negative shear as one-fourth of the live load end shear V_{ba}. From the preceding calculations, the live load end shears for a uniform load of 4 kips per foot are:

$$\text{At end } a \quad \left(\frac{4}{6}\right)(38.9) \quad = \quad 26.0 \quad \therefore +V_{\mathcal{C}} = \frac{26.0}{4} = 6.5$$

$$\text{At end } b \quad \left(\frac{4}{6}\right)(-45.1) = -30.0 \quad \therefore -V_{\mathcal{C}} = \frac{-30.0}{4} = -7.5$$

As the dead load end shear V_{ab} is 13.0 kips, the dead load shear at the center is

$$13.0 - (2)(7.0) = -1.0 \text{ kips}$$

As the shear at the center of span *ab* for loads on the other spans *bc* and *cd* is the same as for the end shears V_{ab} and V_{ba}, the combined values give:

$$V_{\mathcal{C}} = 6.5 - 1.0 - 1.8 + 0.9 = 4.6 \quad (3.2 \text{ by exact analysis})$$

$$V_{\mathcal{C}} = -7.5 - 1.0 - 5.5 + 0.3 = -13.7 \quad (-13.5 \text{ by exact analysis})$$

The typical maximum shear curves illustrated in Fig. 57*a* show that a straight line is a close approximation to the actual curve, and, therefore, a linear variation will be used for this problem (see Fig. 57*b*).

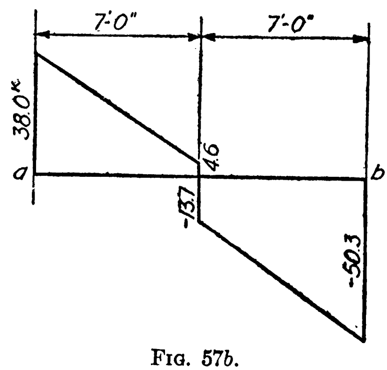

FIG. 57*b*.

30. Summary. The preceding discussion of the analysis of building frames under the action of vertical loads is concerned with a general method of solution rather than a detailed procedure for design calculations. The application of this method of analysis to any building frame will naturally require a careful study of some of the following factors:

(*a*) The distribution of the load in any bay to the various two-dimensional frames into which the building is divided. This distribution de-

pends upon the bending and torsional resistance of the floor and its relation to the members of the various frames. The recommendations of the various building codes should be studied with respect to this problem.

(b) A selection of relative $\frac{I}{L}$ values based upon an estimate of the critical conditions for various members. In the preliminary design, it should be noted that the shear in the beams and the direct stress in the columns will not be affected greatly by the end moments. If haunched members are used, more difficulty may be encountered in making a preliminary design, particularly for built-up steel sections. For reinforced-concrete structures the recommendations of the A.C.I. Building Code on assumptions for design should be studied.

(c) Any unusual conditions should be noted and provided for in the preliminary design by means of an approximate analysis or from experience.

(d) The designer should always remember that the moments that are computed are but one factor in the design, and that only reasonable accuracy in their determination is essential. A considerable variation in the stiffness of a member is required to produce an appreciable change in the moments. The designer must be able to recognize in advance the relative importance of various factors.

PROBLEMS

37. Calculate the end moments in the roof beams and supporting columns of the frame in Fig. 49 for a uniform load of 1 kip per linear foot on span *ef*. Do the same for a unit uniform load on spans *fg* and *gh*, respectively. Draw the bending-moment diagrams for each loading, and establish the positions of the fixed points.

38. From the results of Problem 37, prepare maximum and minimum moment curves for spans *ef* and *fg* for a dead load of 2 kips per foot and a live load of 3 kips per foot.

From the data given in Figs. 52 and 53 together with your results from Problem 37, draw the maximum and minimum moment curves for columns *ae* and *bf*.

39. Calculate the maximum end shears for spans *ef* and *fg* for the same dead and live loads as in Problem 38. By the method followed in Example 19, draw approximate maximum shear curves.

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CHAPTER V

CONTINUOUS FRAMES WITH JOINTS HAVING DIFFERENT LINEAR DISPLACEMENTS

The Use of Auxiliary Force Systems to Control Translation of the Joints

31. Nature of Auxiliary Forces. When the joints of continuous frame structures undergo motion of translation as well as rotation, direct methods of successive approximation are not easily applied as the increments may then converge slowly if at all. Failure to obtain convergence is largely due to the fact that the translation of any joint will have an important effect upon the end moments of many members whereas the rotation of a joint affects primarily the contiguous members. For this reason the analysis of many continuous frames by methods of successive approximation, such as the moment-distribution method, is practical only if the translation of the joints is controlled by the use of auxiliary force systems. The nature of such auxiliary force systems will necessarily depend upon the motion that they must give or prevent in the structure, but, for any problem, the summation of the auxiliary force systems and their displacements must be equivalent to the actual force system and the actual movement of the frame.

The use of an auxiliary force system in controlling one displacement has already been illustrated by the analysis of the frame in Fig. 45. The application of this method of procedure to frames with several displacements will now be discussed. A common example is the frame in Fig. 58*a*, which is acted upon by forces P_1, P_2, P_3. To prevent any horizontal displacements of the joints it will be necessary to apply a system of auxiliary forces F_b, F_c, F_d, as shown by the dotted lines. The analysis by the moment-distribution method can now proceed in the ordinary manner, and the values of F_b, F_c, and F_d can be calculated. To remove these forces, three different auxiliary force systems, as shown by Figs. 58*b*, *c*, and *d*, will be required, each of which permits but one horizontal displacement. The algebraic relation existing between any single displacement Δ and the force system producing it will then be known if it is determined for any assumed value of Δ, as the relationship is a linear one. For this reason each auxiliary force system and the shears and moments which it produces can be expressed in terms of one displace-

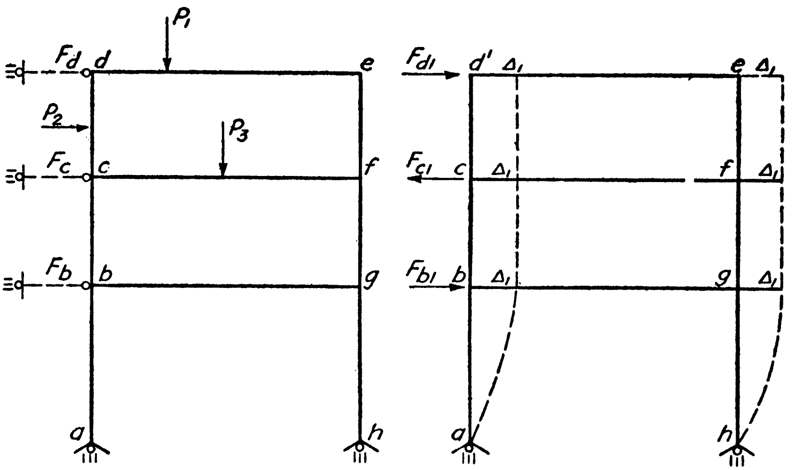


FIG. 58*a* and *b*.

ment Δ, and the summation of all auxiliary force systems can be expressed as functions of the various Δ values. The numerical value of the displacements Δ arc calculated from the necessary equilibrium conditions. As an example of such equilibrium conditions the forces in

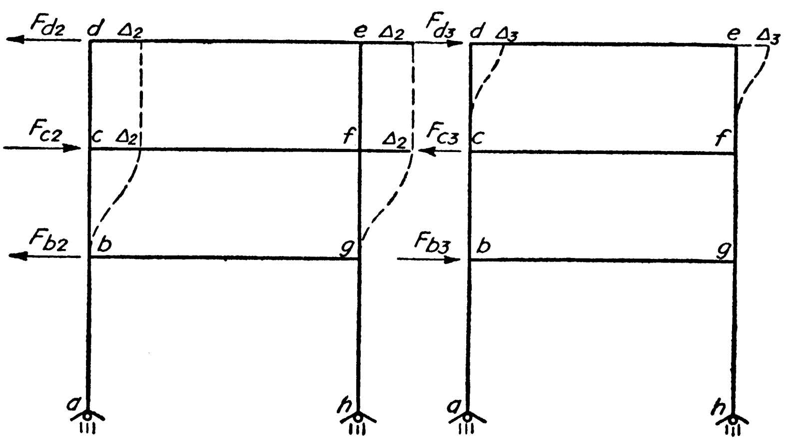


FIG. 58*c* and *d*.

Figs. 58*b*, *c*, and *d*, when combined, must remove the virtual loads F_b, F_c, and F_d used in Fig. 58*a*, or, expressed algebraically,

$$\begin{aligned} F_{b1} + F_{b2} + F_{b3} + F_b &= 0 \\ F_{c1} + F_{c2} + F_{c3} + F_c &= 0 \\ F_{d1} + F_{d2} + F_{d3} + F_d &= 0 \end{aligned} \tag{47}$$

Although these equations are sufficient to determine the true value of Δ_1, Δ_2, and Δ_3, it will be shown later that the numerical calculations are often simplified if the equilibrium conditions are taken with respect to sections through the structure rather than around the joints.

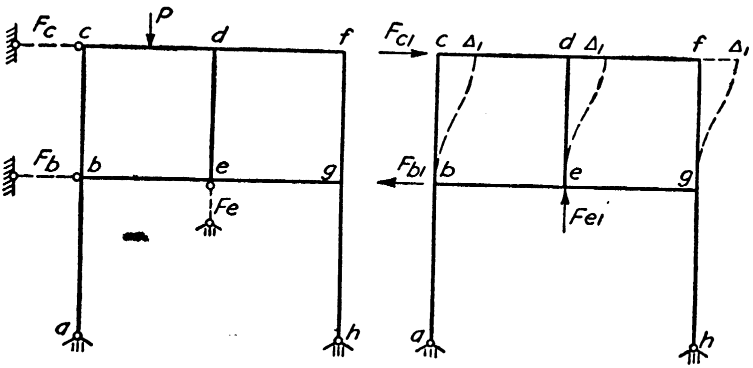


FIG. 59*a* and *b*.

Further examples of auxiliary force systems are given by the frame in Fig. 59*a*, which is subjected to the load P and will require the auxiliary forces F_b, F_c, and F_e to prevent translation of the joints. These forces are removed by combining the auxiliary force systems shown by Figs. 59*b*, *c*, and *d*, each of which is expressed by a single value of Δ. The

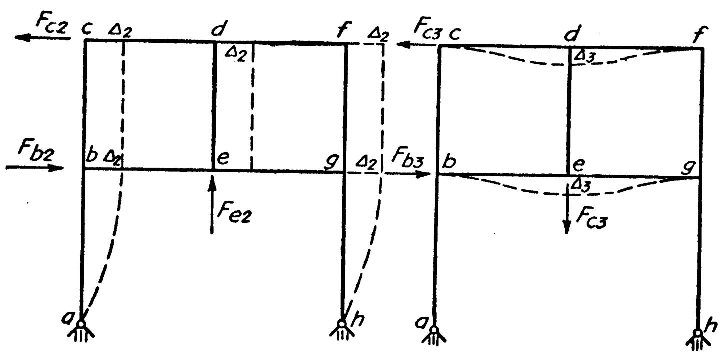


FIG. 59*c* and *d*.

values of Δ_1, Δ_2, and Δ_3 are obtained from the equilibrium conditions that will be explained in a numerical example.

In a similar manner, the frame of Fig. 60*a* can be solved by combining the solutions of the force systems shown in Figs. 60*b*, *c*, and *d*. (See Article 18, Chapter II, for the motion of joint *c*.)

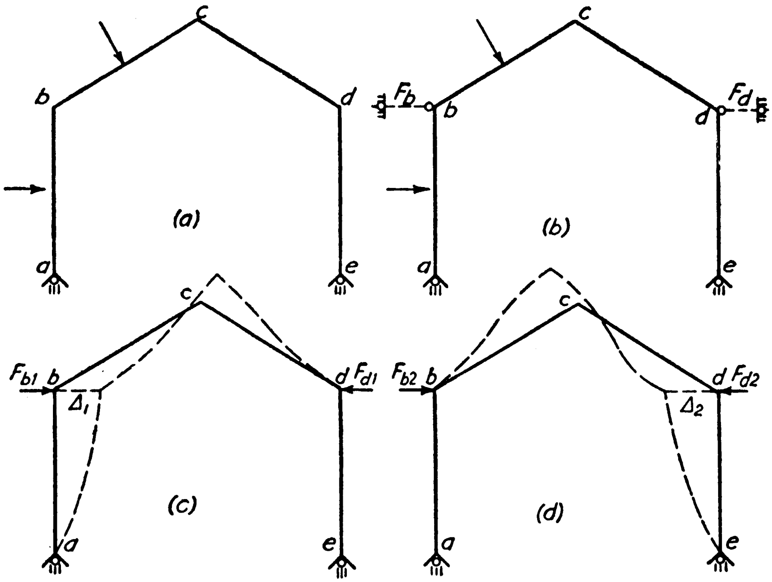


FIG. 60.

Example 20. The above procedure for controlling the translation of the joints will be followed in solving for the moments in the frame of Fig. 61. The solution can be made by means of the moment-distribution method if the horizontal movement of the structure is considered in terms of three separate quantities, Δ_1, Δ_2, and Δ_3, as illustrated in Figs. 58*b*, *c*, and *d*. Let us first assume that $E\Delta_1$ is equal to 10 kips per foot and all rotations are zero except at the supports *a* and *a'*. Then, the only members subjected to flexure are *ab* and *a'b'*, which will be acted upon by fixed-end moments at *b* and *b'* equal to (see equation 26, Article 20)

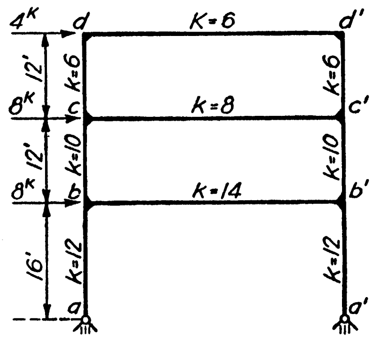


FIG. 61.

$$M_{Fba} = M_{Fb'a'} = -\frac{3EI}{L}\frac{\Delta_1}{L} = -\frac{(3)(12)(10)}{16} = -22.5 \text{ ft-kips}$$

(The value of K is also taken in foot units, although this assumption is not essential if the actual motion is not desired.)

The fixed-end moments due to the translation Δ can also be represented by the term M_F as these moments are always kept separate from those due to transverse loads on the members. These fixed-end moments that are due to an assumed value of Δ_1 are corrected for rotation of the joints by the moment-distribution method as indicated by the numerical operation in Fig. 62. In this problem considerable numerical

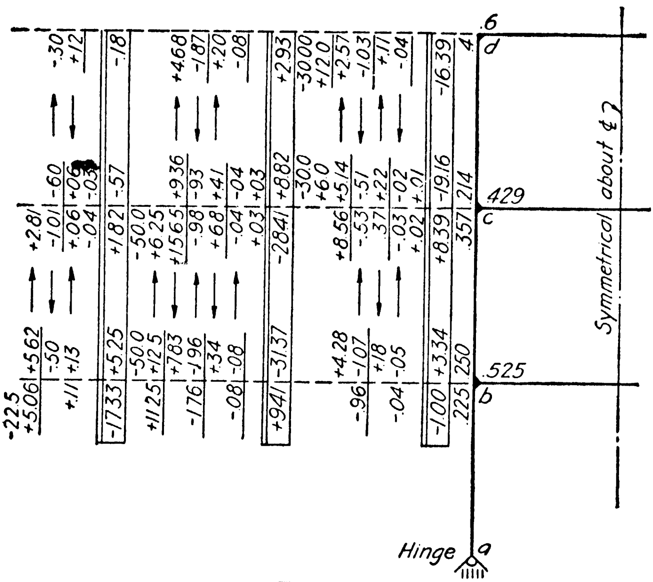


FIG. 62.

work is saved by taking advantage of the symmetry of the structure. The slope-deflection equation for the moment $M_{bb'}$, which is

$$M_{bb'} = (4)(14)\theta_b + (2)(14)\theta_{b'}$$

can be written

$$M_{bb'} = (6)(14)\theta_b = 84\theta_b$$

since by symmetry we know that θ_b is equal to $\theta_{b'}$. Therefore, when joint b is permitted to rotate an amount θ_b that is necessary to balance the moments around the joint, joint b' is assumed to rotate the same amount. This means that joints b and b' are rotated at the same time. Joints a and a' are free to rotate while the remaining joints are kept in a fixed condition. The moments around the joint b due to any rotation θ_b and $\theta_{b'}$ will therefore be

$$M_{ba} = (3)(12)\theta_b = 36\theta_b \qquad M_{bb'} = 84\theta_b \qquad M_{bc} = (4)(10)\theta_b = 40\theta_b$$

The distribution factors at joint b will be

$$r_{ba} = \frac{36}{160} = 0.225 \qquad r_{bb'} = \frac{84}{160} = 0.525 \qquad r_{bc} = \frac{40}{160} = 0.25$$

In the same manner, the distribution factors at joint c can be computed:

$$r_{cb} = \frac{40}{112} = 0.357 \qquad r_{cc'} = \frac{48}{112} = 0.429 \qquad r_{cd} = \frac{24}{112} = 0.214$$

and, at joint d:

$$r_{dc} = \frac{24}{60} = 0.4 \qquad r_{dd'} = \frac{36}{60} = 0.6$$

The distribution of the fixed-end moments is carried out in the usual manner as indicated by the arrows in Fig. 62. After the end moments have been determined, the value of the shear in each panel and the value of the auxiliary forces can be calculated. Thus, the value of the shear, due to Δ_1, is

Panel 1 $\quad V_1 = \dfrac{(2)(17.33)}{16} = 2.17 \quad \text{or} \quad 0.217\Delta_1 \rightarrow$

Panel 2 $\quad V_2 = -\dfrac{(2)(5.25 + 1.82)}{12} = -1.18 \quad \text{or} \quad -0.118\Delta_1 \leftarrow$

Panel 3 $\quad V_3 = \dfrac{(2)(0.75)}{12} = 0.125 \quad \text{or} \quad 0.0125\Delta_1 \rightarrow$

In a similar manner, fixed-end moments of -50 ft-kips in panel 2, due to an assumed movement of $E\Delta_2$ equal to 10, are distributed and the shear in each panel calculated. The shear values will be

$$V_1 = -0.118\Delta_2 \qquad V_2 = 0.995\Delta_2 \qquad V_3 = -0.196\Delta_2$$

For a horizontal movement $E\Delta_3$ equal to 10, fixed-end moments of $-\dfrac{(6)(6)(10)}{12} = -30$ ft-kips will occur in panel 3. After these moments are distributed, the following shears in each panel are obtained.

$$V_1 = 0.0125\Delta_3 \qquad V_2 = -0.196\Delta_3 \qquad V_3 = 0.592\Delta_3$$

If the shears in each panel due to the auxiliary force systems are now set equal to the actual shear of the applied forces, a set of linear equations in terms of Δ_1, Δ_2, and Δ_3 is obtained.

Panel 1 $\quad +0.217\,\Delta_1 - 0.118\Delta_2 + 0.0125\Delta_3 = 20$

Panel 2 $\quad -0.118\,\Delta_1 + 0.995\Delta_2 - 0.196\,\Delta_3 = 12$

Panel 3 $\quad +0.0125\Delta_1 - 0.196\Delta_2 + 0.592\,\Delta_3 = 4$

As the coefficients of these equations are arranged symmetrically with respect to the main diagonal and with the large coefficients on the diagonal, a solution for the Δ values can be made by the method of iteration. This method is of sufficient importance in solving such simultaneous equations to warrant a detailed description of the procedure.

The equations will be rearranged by solving the first for Δ_1, the second for Δ_2, and the third for Δ_3, giving:

$$\Delta_1 = 92.2 + 0.543\Delta_2 - 0.058\Delta_3$$

$$\Delta_2 = 12.1 + 0.119\Delta_1 + 0.197\Delta_3$$

$$\Delta_3 = 6.75 - 0.021\Delta_1 + 0.331\Delta_2$$

If we first assume that Δ_2 and Δ_3 are zero, a value of 92.2 is obtained as a first trial value for Δ_1. This value of Δ_1 is then put in the next equation for Δ_2, which gives

$$\Delta_2 = 12.1 + (0.119)(92.2) + 0 = 23.0$$

These values of Δ_1 and Δ_2 are then inserted in the equation for Δ_3, giving

$$\Delta_3 = 6.75 - (0.021)(92.2) + (0.331)(23.0) = 12.4$$

A new value of Δ_1 can now be obtained by using the above values of Δ_2 and Δ_3,

$$\Delta_1 = 92.2 + (0.543)(23.0) - (0.058)(12.4) = 104.0$$

This procedure is continued until the desired accuracy is obtained. The values are tabulated in Table 3, from which it can be seen that the convergence is rapid. In this case three cycles are sufficient for all practical purposes.

TABLE 3

Cycle	$E\,\Delta_1$	$E\,\Delta_2$	$E\,\Delta_3$
1	92.2	23.0	12.4
2	104.0	26.8	13.44
3	106.0	27.2	13.52
4	106.2	27.3	13.55

As the moments in Fig. 62 are for $E\Delta$ values of 10, the moments for the above values of $E\Delta$ can be obtained by proportion, thus

$$M_{ba} = (-1.733)(106.2) + (0.941)(27.3) - (0.100)(13.55) = -159.8$$

$$M_{bc} = (0.525)(106.2) - (3.137)(27.3) + (0.334)(13.55) = -25.4$$

$$M_{cb} = -46.8 \qquad M_{cd} = -8.0 \qquad M_{dc} = -16.1$$

Example 21. The frame shown in Fig. 63a will be analyzed by the procedure that has already been discussed for Fig. 59. Here the vertical load of 21 kips that is applied on the member cd will produce a rotation and displacement at each joint. If the change in length of the members is neglected, three different linear displacements, two horizontal and one vertical, must be considered. These displacements can

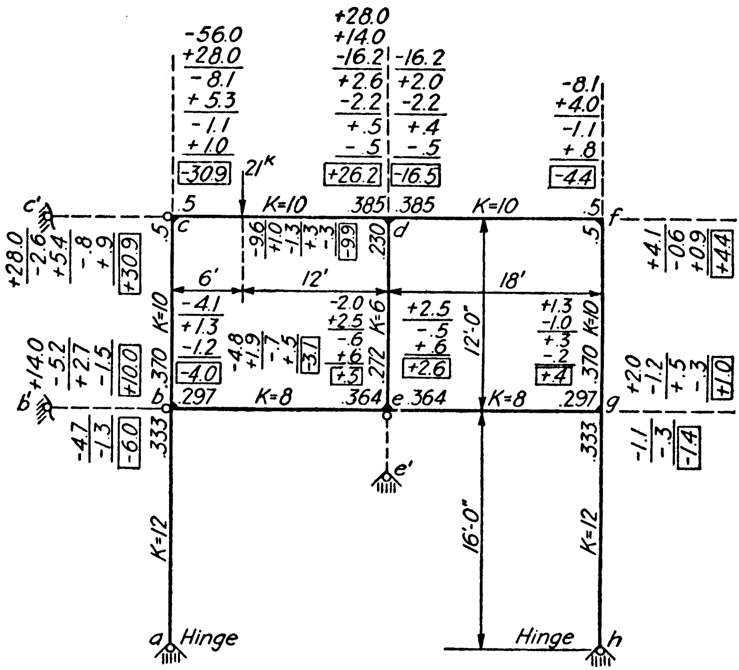


FIG. 63a.

be prevented by applying auxiliary forces at b, c, and e as illustrated in Fig. 63a. When the frame is held in this position, the fixed-end moments

$$M_{Fcd} = -56.0 \text{ ft-kips}$$

and

$$M_{Fdc} = +28.0 \text{ ft-kips}$$

due to the concentrated load of 21 kips, are distributed in the usual manner. The distribution procedure as well as the final moments are given on the diagram. After the end moments have been determined the end shears in each member are calculated and from these values the magnitude of each auxiliary force is obtained. For instance, the force

cc' must be equal to the algebraic sum of the shears in members bc, de, and fg, or

$$cc' = \frac{30.9 + 10.0 - 9.9 - 3.1 + 4.4 + 1.0}{12} = 2.78 \text{ kips} \leftarrow$$

Similarly,

$$bb' = \frac{-6.0 - 1.4}{16} - 2.78 = -3.24 \text{ kips} \rightarrow$$

$$ee' = V_{dc} + V_{df} + V_{eb} + V_{eg} = +6.74 + 1.16 - 0.19 - 0.17$$
$$= 7.54 \text{ kips} \uparrow$$

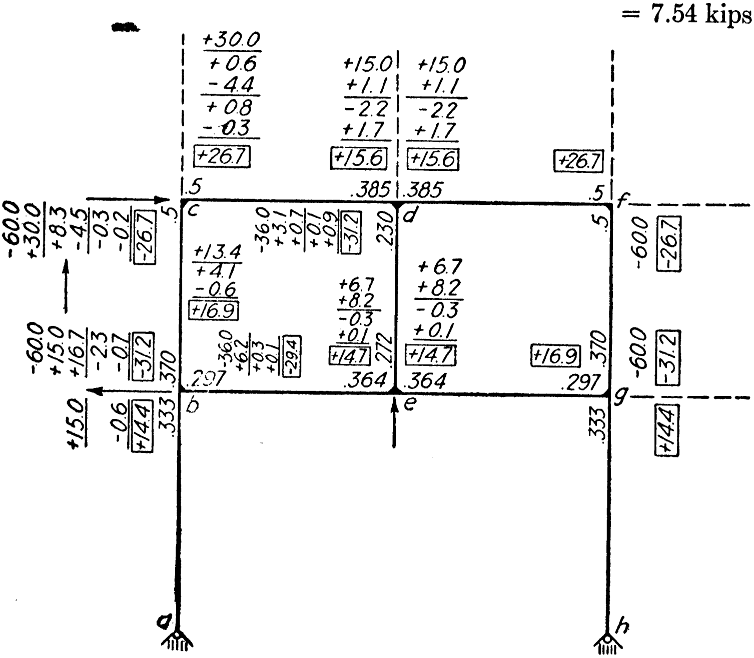


Fig. 63b.

To remove these auxiliary forces, the procedure already explained for Figs. 59b, c, and d is used, i.e., giving the structure three separate displacements Δ_1, Δ_2, and Δ_3. Let us first assume $\Delta_1 = 12$ units to the right; then

$$M_{Fbc} = M_{Fcb} = -\frac{(6)(10)(12)}{12} = -60$$

$$M_{Fed} = M_{Fde} = -\frac{(6)(6)(12)}{12} = -36$$

In the distribution of these moments (Fig. 63*b*) advantage can be taken of the symmetry of the structure to reduce the numerical work.

For Δ_1 equal to 12 units, the auxiliary forces are

$$cc' = \frac{(-26.7 - 31.2)(2) - 31.2 - 29.4}{12} = -14.68 \quad \text{or} \quad 1.222\Delta_1 \rightarrow$$

$$bb' = \frac{14.4 + 14.4}{16} + 14.68 = 16.48 \quad \text{or} \quad 1.374\Delta_1 \leftarrow$$

$$ee' = 0$$

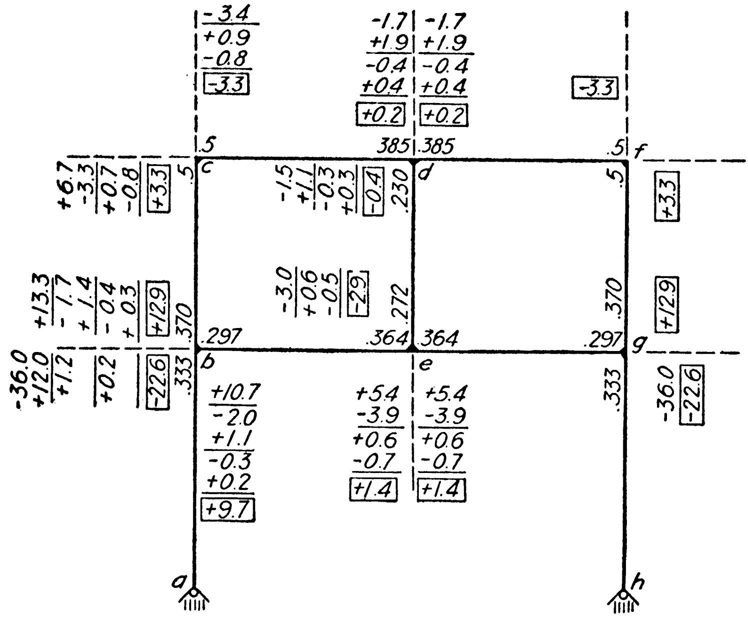


FIG. 63*c*.

If a value of Δ_2 equal to 16 units is assumed (see Fig. 63*c*), and if the joints are held without rotation, there will be fixed-end moments in members *ab* and *gh* equal to

$$M_{Fba} = M_{Fgh} = -\frac{(3)(12)(16)}{16} = -36$$

The joints are now permitted to rotate in the following order: *b*, *c*, *g*, *f*, *e*, *d*. It will be found that, owing to symmetry, only the joints *b*, *c*, *d*, and *e* need be used in the distribution of the moments. The numerical operations and the final moments are given on the diagram.

The auxiliary forces in terms of Δ_2 are

$$cc' = \frac{2(12.9 + 3.3) - 2.9 - 0.4}{12} = 2.43 \quad \text{or} \quad 0.1518\Delta_2 \leftarrow$$

$$bb' = \frac{-22.6 - 22.6}{16} - 2.43 = -5.26 \quad \text{or} \quad 0.329\Delta_2 \rightarrow$$

$$ee' = 0$$

For a value of Δ_3 (see Fig. 59*d*) equal to 18 the fixed-end moments are

$$M_{Fcd} = M_{Fdc} = -\frac{(6)(10)(18)}{18} = -60 \qquad M_{Fdf} = M_{Ffd} = +60$$

$$M_{Fbe} = M_{Feb} = -\frac{(6)(8)(18)}{18} = -48 \qquad M_{Feg} = M_{Fge} = +48$$

The distribution of these fixed-end moments, as well as the final value of the end moments, is given in Fig. 63*d*. The auxiliary forces required to produce these moments are

$$cc' = 0 \qquad bb' = 0 \qquad ee' = \frac{2(33.2 + 46.6 + 37.8 + 42.9)}{18} = 17.84$$

$$\text{or} \quad 0.99\Delta_3 \downarrow$$

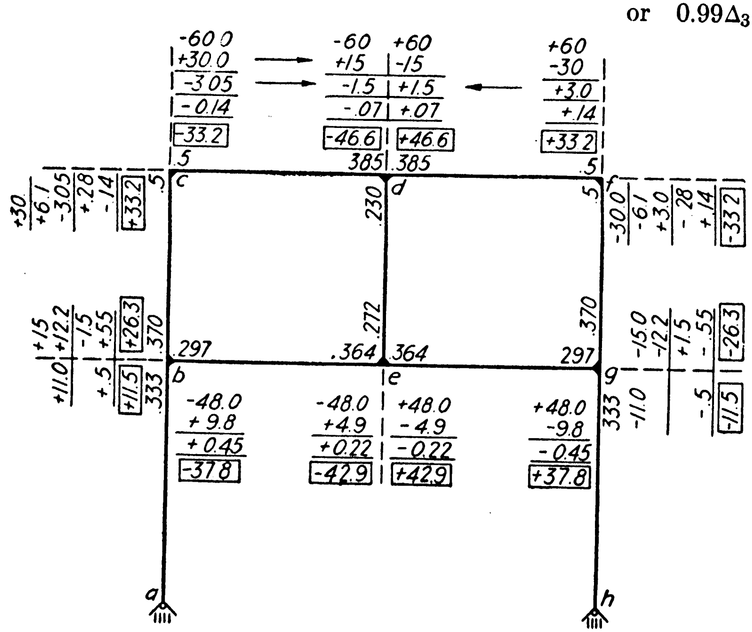


FIG. 63*d*.

To determine the values of Δ_1, Δ_2, and Δ_3 that will reduce the auxiliary force system to zero, the following equilibrium equations are necessary.

Horizontal forces at c must equal zero, or

$$(a) \quad 1.222\Delta_1 - 0.1518\Delta_2 - 2.78 = 0$$

Horizontal shear in first story must be zero, or

$$(b) \quad 1.222\Delta_1 - 1.374\Delta_1 - 0.1518\Delta_2 + 0.329\Delta_2 - 2.78 + 3.24 = 0$$

which gives $-0.152\Delta_1 + 0.1772\Delta_2 = -0.46$.

Forces at e must equal zero

$$(c) \quad -0.99\Delta_3 + 7.54 = 0$$

From (c)

$$\Delta_3 = \frac{7.54}{0.99} = 7.62$$

From (a) and (b)

$$\Delta_1 = 2.185 \quad \Delta_2 = -0.723$$

The final moments are obtained by multiplying the moments in Fig. 63b by $\frac{2.185}{12}$, in Fig. 63c by $-\frac{0.723}{16}$, and in Fig. 63d by $\frac{7.62}{18}$, and then by combining these values with the moments of Fig. 63a. The sum of the four sets of moments (Fig. 63e) gives the actual values.

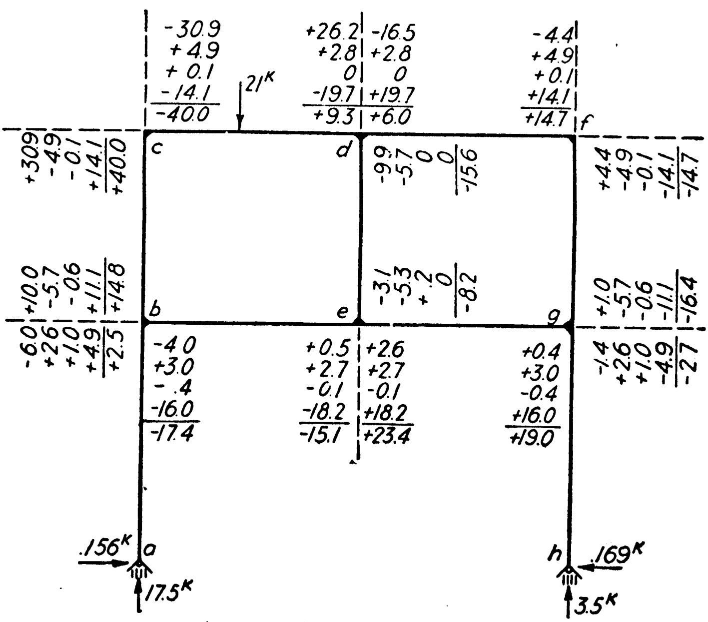


FIG. 63e.

32. Frames with Inclined Members. The roof frame of Fig. 60a is one example of a continuous frame in which the axes of the various members do not intersect at right angles. Many similar structures can be found in practice, all of which can be analyzed by the preceding methods of solution. The geometrical relations that exist between the linear displacements of the joints for the assumption that the lengths of the members do not change must always be carefully established. As the construction and application of the Williot diagram for solving this problem are explained in Article 18, Chapter II, it will now be used directly in the solution of specific structures.

The bent in Fig. 64a, which is a type of structure that is common in viaducts, must undergo the displacements shown by the dotted lines. The relative magnitudes

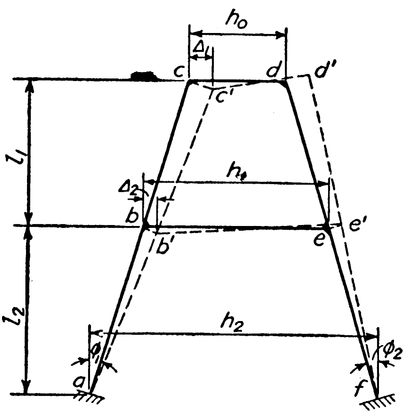


FIG. 64a.

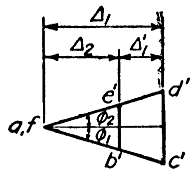


FIG. 64b.

of these displacements in terms of Δ_1 and Δ_2, the horizontal displacements, can be readily obtained from the displacement diagram in Fig. 64b. The rotations of the axes of the various members are

$$
\begin{aligned}
&\text{Member } ab \qquad \frac{\Delta_{ab}}{ab} = \frac{\Delta_2}{\cos\phi_1}\cdot\frac{\cos\phi_1}{l_2} = \frac{\Delta_2}{l_2} \\
&\text{Member } ef \qquad \frac{\Delta_{ef}}{ef} = \frac{\Delta_2}{\cos\phi_2}\cdot\frac{\cos\phi_2}{l_2} = \frac{\Delta_2}{l_2} \\
&\text{Member } be \qquad \frac{\Delta_{be}}{be} = \frac{\Delta_2(\tan\phi_1+\tan\phi_2)}{h_1} = \frac{\Delta_2}{l_2}\,\frac{(h_2-h_1)}{h_1} \\
&\text{Member } bc \qquad \frac{\Delta_{bc}}{bc} = \frac{\Delta_1-\Delta_2}{\cos\phi_1}\cdot\frac{\cos\phi_1}{l_1} = \frac{\Delta_1-\Delta_2}{l_1} = \frac{\Delta_1'}{l_1} \\
&\text{Member } de \qquad \frac{\Delta_{de}}{de} = \frac{\Delta_1-\Delta_2}{\cos\phi_2}\cdot\frac{\cos\phi_2}{l_1} = \frac{\Delta_1-\Delta_2}{l_1} = \frac{\Delta_1'}{l_1} \\
&\text{Member } cd \qquad \frac{\Delta_{cd}}{cd} = \frac{\Delta_1(\tan\phi_1+\tan\phi_2)}{h_0} = \frac{(\Delta_1'+\Delta_2)}{l_1}\,\frac{(h_1-h_0)}{h_0}
\end{aligned}
\tag{48}
$$

In the solution of any numerical problem, the unknown moments and forces are taken either in terms of the relative horizontal displacement Δ' in each panel or the actual displacements Δ_1 and Δ_2. The numerical calculations are based on the use of auxiliary force systems as in the preceding problems.

Example 22. In the analysis of the frame in Fig. 65*a* only half of the structure need be considered in the computations because of symmetry.

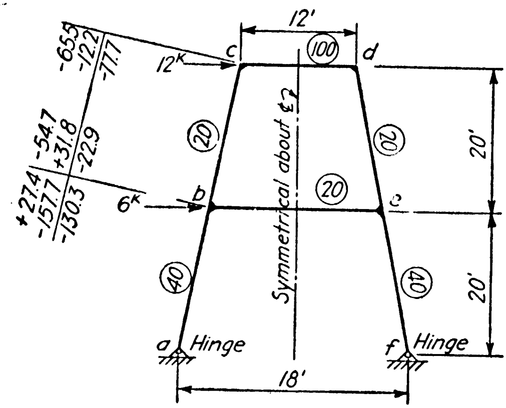


FIG. 65*a*.

Thus, it is apparent that $\theta_d = \theta_c$ and $\theta_e = \theta_b$, and, consequently, the moments at ends of members *cd* and *be* are equal to $6K\theta_c$ and $6K\theta_b$, respectively, when only joint rotations are considered. For this condition, the distribution factors at joints *b* and *c* are equal to

JOINT *b*

Member *ba*	$(3)(40)\theta_b = 120\theta_b$	$r_{ba} = 0.375$
Member *be*	$(6)(20)\theta_b = 120\theta_b$	$r_{be} = 0.375$
Member *bc*	$(4)(20)\theta_b = 80\theta_b$	$r_{bc} = 0.250$
	$320\theta_b$	1.000

JOINT *c*

Member *bc*	$(4)(20)\theta_c = 80\theta_c$	$r_{bc} = 0.118$
Member *cd*	$(6)(100)\theta_c = 600\theta_c$	$r_{cd} = 0.882$
	$680\theta_c$	1.000

Let us first assume an auxiliary force system P'_1 and P'_2, together with the reactions, that will produce a relative horizontal movement between joints *b* and *c* of Δ'_1 equal to 10 units with Δ_2 equal to zero.

For this displacement the fixed-end moments (see equations 48) necessary to prevent rotation of the joints are

$$M_{Fba} = 0 \qquad M_{Fbe} = 0 \qquad M_{Fbc} = M_{Fcb} = -\frac{(6)(20)(10)}{20} = -60$$

$$M_{Fcd} = +(6)(100)\frac{(10)}{20}\left(\frac{15-12}{12}\right) = +75$$

The final moments and the horizontal and vertical components of the end forces are given in Fig. 65*b*. It should be noted that the shearing

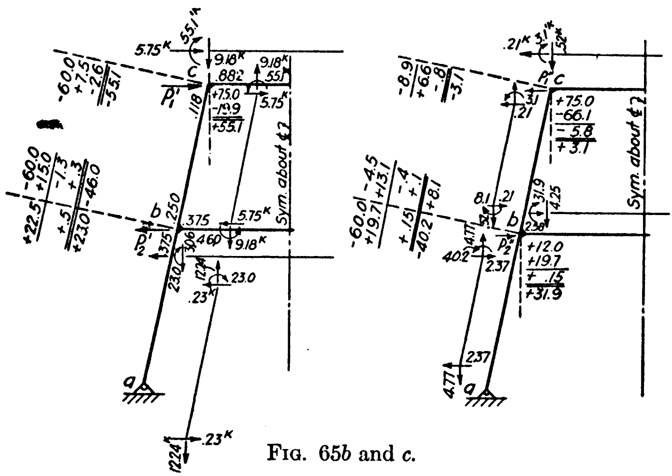


FIG. 65*b* and *c*.

forces in the horizontal members *cd* and *bc* must be calculated before the vertical and horizontal forces in the columns can be obtained. The values of P_1' and P_2' are equal to

$$P_1' = (2)(5.75) = 11.50 \qquad P_2' = (2)(5.75 + 0.23) = 11.96$$

or, in terms of Δ_1',

$$P_1' = 1.150\Delta_1' \rightarrow \qquad P_2' = 1.196\Delta_1' \leftarrow$$

In the same manner, the auxiliary force system necessary to give a relative horizontal movement between *b* and *a* is determined. Let us assume a value of Δ_2 equal to 10 units and Δ_1' equal to zero. Then

$$M_{Fba} = -\frac{(3)(40)(10)}{20} = -60$$

$$M_{Fbe} = +(6)(20)\frac{(10)}{20}\frac{(3)}{15} = +12$$

$$M_{Fcd} = +\frac{(6)(100)(10)}{20}\frac{(3)}{12} = +75$$

From the moments and forces given in Fig. 65c the auxiliary force system for any displacement Δ_2 is found to be

$$P_1'' = 0.042\Delta_2 \leftarrow \qquad P_2'' = 0.516\Delta_2 \rightarrow$$

The actual values of Δ_1' and Δ_2 are obtained from the equilibrium conditions that

$$(a) \quad P_1' + P_1'' = 12.0$$

or

$$1.150\Delta_1' - 0.042\Delta_2 = 12.0$$

$$(b) \quad P_2' + P_2'' = 6.0$$

or

$$-1.196\Delta_1' + 0.516\Delta_2 = 6.0$$

from which $\Delta_1' = 11.9$, $\Delta_2 = 39.2$.

The final moments given in Fig. 65a are obtained by multiplying the moments in Fig. 65b by 1.19 and those in Fig. 65c by 3.92.

Example 23. The ridge or gable frame illustrated in Fig. 66a can also be solved by the foregoing procedure with the aid of the geometrical

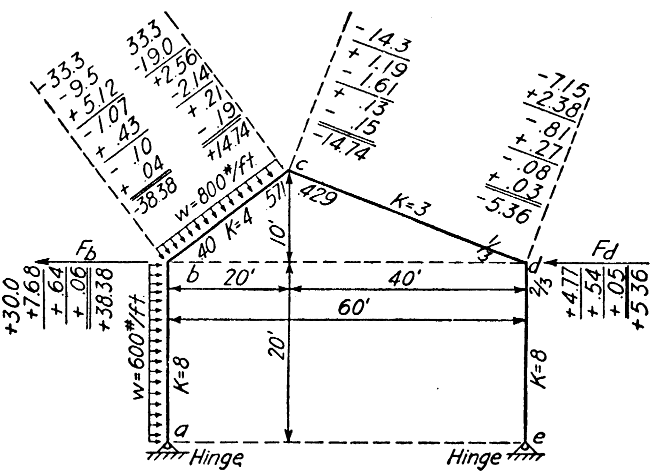


FIG. 66a.

relations derived in Article 18, Chapter II, for this type of structure. As the translation of joint c is dependent upon the movement of joints b and d, two auxiliary forces, F_b and F_d, will be sufficient to prevent translation of all joints. The fixed-end moments due to the uniform loads on members ab and bc are distributed in the usual manner. With the end moments known, the horizontal components at the ends of each mem-

ber can be determined without difficulty. Thus, in members *ab* and *de*,

$$H_{ba} = \frac{(0.600)(20)(10) + 38.38}{20} = 7.919 \leftarrow \qquad H_{de} = \frac{5.36}{20} = 0.268 \leftarrow$$

and, considering the portion *bcd*,

$$V_{dc} = \frac{(16.0)(10) + (8)(5) - 38.38 - 5.36}{60} = 2.604 \uparrow$$

$$V_{bc} = 16.0 - 2.604 = 13.396 \uparrow$$

By taking moments about *c* for forces on member *bc*

$$H_{bc} = \frac{-200 - 38.38 + 14.74 + (13.396)(20)}{10} = 4.428 \rightarrow$$

for member *cd*

$$H_{dc} = \frac{(2.604)(40) + 5.36 + 14.74}{10} = 12.426 \leftarrow$$

The auxiliary forces F_b and F_d will therefore be

$$F_b = 7.919 - 4.428 = 3.491$$

$$F_d = 12.426 + 0.268 = 12.694$$

Forces F_b and F_d must now be removed by superimposing the force systems required to produce the separate displacements Δ_b and Δ_d (see Figs. 60*c* and *d*). If joint *b* is assumed to move to the right an amount Δ_b equal to 10 and Δ_d equal zero, then, by the relations given in equations 22 and 23, Chapter II,

$$\frac{\Delta_b - \Delta_a}{ab} = \frac{10}{20} \qquad \frac{\Delta_c - \Delta_b}{bc} = \frac{(10)(40)}{(10)(60)} = \frac{2}{3}$$

$$\frac{\Delta_c - \Delta_d}{cd} = \frac{(10)(20)}{(10)(60)} = \frac{1}{3}$$

From the above relations the fixed-end moments for the various members for Δ_b equal to 10 are

$$M_{Fba} = -(3)(8)\left(\frac{10}{20}\right) = -12.0 \qquad M_{Fbc} = M_{Fcb} = +(6)(4)\frac{2}{3} = +16.0$$

$$M_{Fcd} = M_{Fdc} = -(6)(3)\left(\tfrac{1}{3}\right) = -6.0$$

The correction of these fixed-end moments for rotation of the joints and the auxiliary forces F'_b and F'_d necessary to hold the structure are given in Fig. 66*b*.

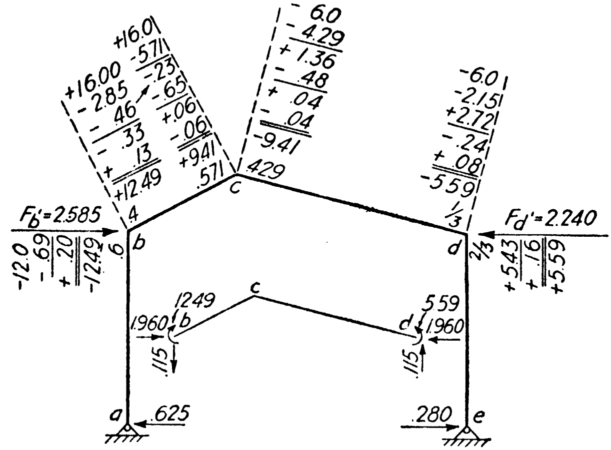


FIG. 66*b*.

If joint d is now assumed to move to the right an amount Δ_d equal to 10 while joint b is prevented from moving, the fixed-end moments in the various members are

$$M_{Fbc} = M_{Fcb} = -\frac{(6)(4)(10)(40)}{(10)(60)} = -16.0$$

$$M_{Fcd} = M_{Fdc} = +\frac{(6)(3)(10)(20)}{(10)(60)} = +6.0$$

$$M_{Fde} = -(3)(8)\left(\frac{10}{20}\right) = -12.0$$

The moments and auxiliary forces that will produce such a displacement of the frame are given in Fig. 66*c*. From the conditions that

$$F'_b + F''_b + F_b = 0$$

and

$$F'_d + F''_d + F_d = 0$$

the following equations are obtained:

$$0.2585\Delta_b - 0.224\Delta_d - 3.491 = 0$$

$$-0.2240\Delta_b + 0.2261\Delta_d - 12.694 = 0$$

from which $\Delta_b = 439.4$, $\Delta_d = 491.6$.

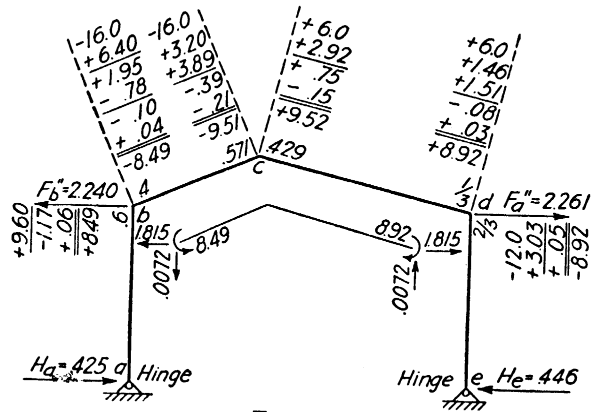


FIG. 66c.

The resultant moments (Fig. 66*d*) are obtained by combining the moments in Fig. 66*a* with 43.94 times those in Fig. 66*b* and 49.16 times those in Fig. 66*c*.

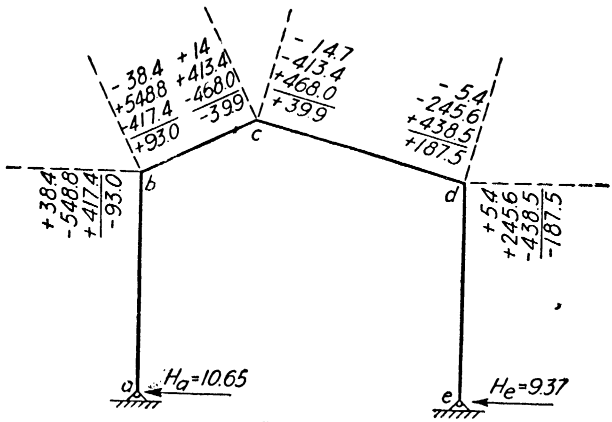


FIG. 66*d*.

33. Advantages and Disadvantages of Auxiliary Force Systems. The method of solution for continuous frames in which the translation of the joints is controlled by auxiliary force systems has certain advantages and disadvantages that will be summarized at this time.

(*a*) For building frames of many stories and bays that are subjected to horizontal loads, the above procedure can be conveniently applied to any part of the structure. The distribution of the fixed-end moments for a relative displacement in any story will extend for only one or two stories on each side. For this reason the solution of the shear equations

by the method of iteration, as in Example 20, can be performed quickly with a slide rule. The relative stiffness of different frames is also determined at the same time, a factor that is of assistance in the allocation of the total horizontal load to the various frames.

(*b*) After the moments have been obtained for one loading condition, solutions for other types of loading can be quickly made. The only difference will be in the constants for evaluating the Δ values.

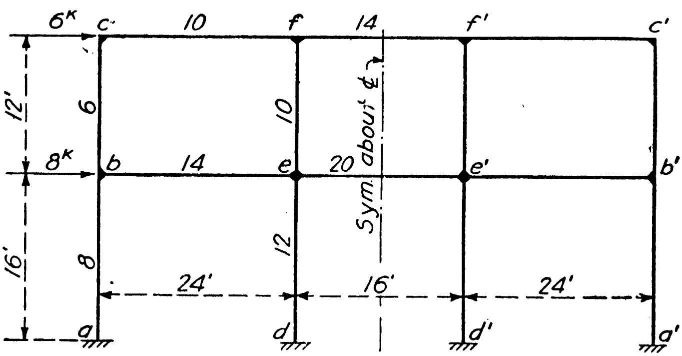
(*c*) In the following chapter it will be shown that the method is also applicable to frames whose members have variable moments of inertia, once the proper coefficients have been obtained.

(*d*) In structures such as the Vierendeel truss with inclined chords, the numerical work is often laborious although less so than for many other methods. The panel method, explained in the following section, is recommended when it is applicable.

(*e*) The moments in certain frames, such as in Example 23, are obtained from the differences of large numbers, and, therefore, in such problems, the Δ values must be computed with considerable accuracy.

PROBLEMS

40. Calculate the end moments in the two-story frame by the use of auxiliary forces and assumed displacements. The condition of symmetry should be utilized in the solution. Compare the results with those obtained for the



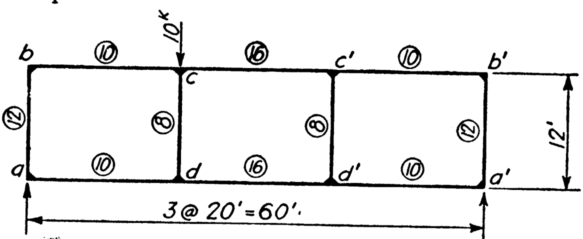
PROBLEM 40.

assumption that the points of contraflexure of beams and columns are at the center of the members and that the inside columns take twice as much shear as the outside columns.

41. Compute the moments in the Vierendeel truss shown. Note that three relative vertical displacements can be used as shown by a displacement diagram that is started with member *ab* assumed vertical; or two vertical displacements

and one horizontal displacement can be used if points a and a' are kept at the same elevation. The choice lies in whether to use the Williot diagram direct or to correct for rotation. The end moments are the same in either case.

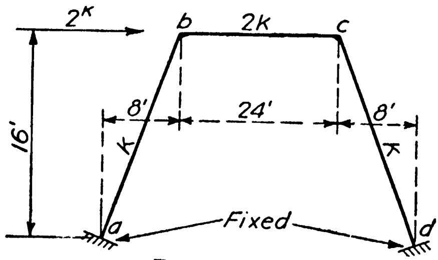
Ans. $M_{bc} = -30.9$ ft-kips; $M_{cb} = -35.8$ ft-kips; $M_{cc'} = +26.9$ ft-kips.



PROBLEM 41.

42. Compute the end moments for the frame shown.

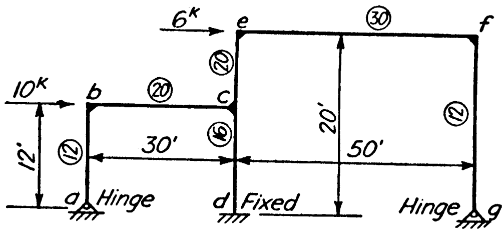
Ans. $M_{ab} = -5.85$ ft-kips; $M_{ba} = -6.08$ ft-kips.



PROBLEM 42.

43. Determine the end moments for all members of the frame shown.

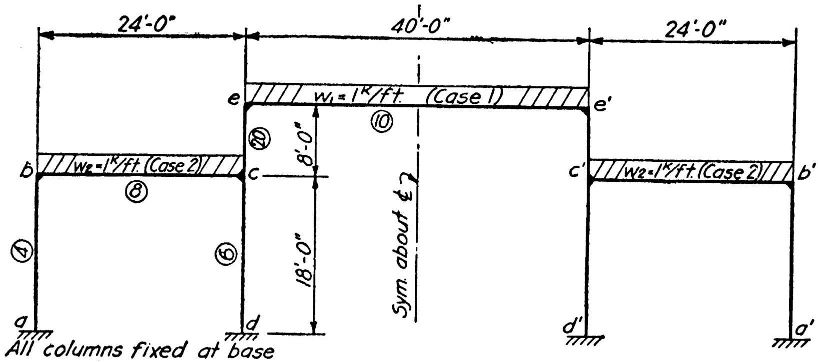
Ans. $M_{ba} = -36.8$ ft-kips; $M_{cb} = +64.0$ ft-kips; $M_{ce} = -6.9$ ft-kips.



PROBLEM 43.

44. Calculate the end moments for the frame shown when there is a weight of 1 kip per foot on the roof of the center bay (no load on side bays). Note: Make use of symmetry in your calculations.

Ans. $M_{ee'} = -71.4$ ft-kips; $M_{ce} = -34.9$ ft-kips; $M_{cd} = +31.5$ ft-kips.



PROBLEM 44.

45. Calculate the end moments for the frame in Problem 44 for a load of 1 kip per foot on both side spans and no weight on center bay. Make use of the symmetry of the frame.

Ans. $M_{ee'} = -6.0$ ft-kips; $M_{ce} = -16.3$ ft-kips; $M_{cd} = -27.6$ ft-kips; $M_{cb} = +43.7$ ft-kips.

Panel Method for Analyzing Quadrangular Frames

34. Historical Development. In 1904, Professor L. F. Nicolai presented a solution, for a parallel-chord Vierendeel truss, based on the assumption that the rectangular panel, composed of the upper and lower chords and the two verticals, is the fundamental structural unit and that the continuity with the remainder of the structure can be ignored.

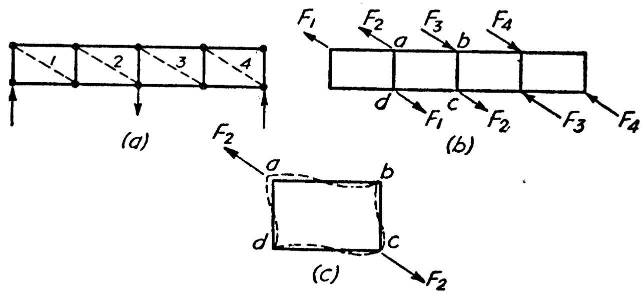


FIG. 67.

Nicolai's analysis was then made by assuming that diagonals are acting in the panels and that the members are pin-connected as shown in Fig. 67*a*. Then at each joint an external force is applied (Fig. 67*b*) that will neutralize the stress in the diagonal. The last step is to consider the panel as a rigid frame acted upon by these forces, as, for example, panel

abcd, Fig. 67*c*. The moments in this frame are assumed to be equal to the moments in the Vierendeel truss. Obviously such a procedure gives only an approximate solution, as the continuity between panels is ignored.

This method, used by Professor Nicolai, was greatly extended and made more practicable in 1921 by K. Čališev, who showed that the verticals of a particular panel could be increased in stiffness until they provided enough additional restraint to compensate for the remainder of the structure. This correction took the form of a rapidly converging series and therefore changed the method from an approximate solution to a solution by successive approximations. Čališev also extended the solution to Vierendeel trusses with inclined chords. However, he still retained the idea of reducing the diagonal stress to zero, which is an unnecessary complication as the forces acting on the panel can be determined without considering imaginary diagonal members at all. The author has therefore modified the procedure by considering the forces acting directly on the panel and by correcting the moments for continuity between the panels rather than by correcting the stiffness of the verticals. These changes make the method analogous to the moment-distribution method except that the action of the panel rather than the joint is the primary factor. The use of the panel permits both rotation and translation of the joints to occur simultaneously and still give a solution by successive approximations that will converge rapidly. The author recommends the use of the panel method for Vierendeel truss systems whenever the chords have approximately the same $\frac{I}{L}$ value, say with a difference less than 15 per cent.

35. Forces Acting on a Panel. If any panel *abcd* of a continuous frame of the Vierendeel truss type, Fig. 68a, in which the chord members *ad* and *bc* have the same K or $\frac{I}{L}$ value, is separated from the structure by passing sections 1–1 and 2–2, the force system acting on the panel will be as shown in Fig. 68*b*. In representing these forces and in the subsequent analysis, the following notation is used:

M = bending moment on section 1–1 = $P_1y_1 + P_2y_2$.

V = shear in panel *abcd* = $P_1 + P_2 + P_3$.

l = panel length.

$K = \frac{I}{L}$ value of chord members *ad* and *bc*.

$K_1, K_2 = \frac{I}{L}$ values of members ab and cd.

$$r = \frac{K}{K_1}, \quad s = \frac{K}{K_2}, \quad \alpha = \frac{h_2 - h_1}{h_1}.$$

$$D = 6 + r + s + \alpha(2\alpha + \alpha s + 2s + 6).$$

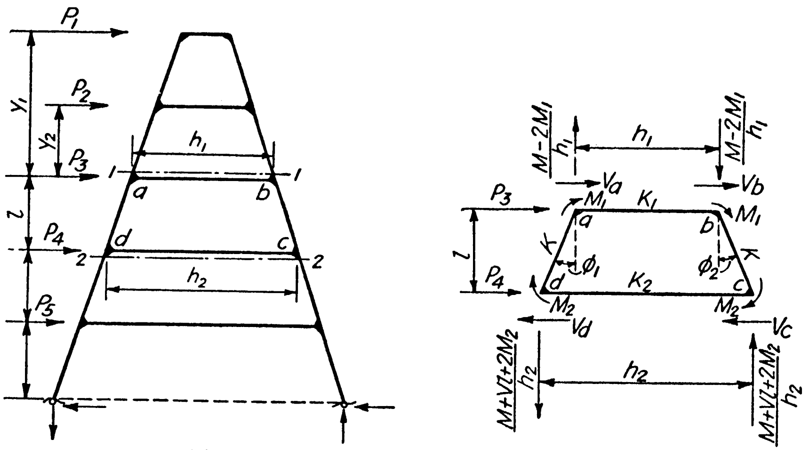


Fig. 68. Forces acting on the panel of a quadrangular frame.

For purposes of analyzing the moments in the panel *abcd*, Fig. 68*b*, the actual force system can be resolved into three equivalent force systems. These three force systems are shown in Figs. 69*a*, *b*, and *c*, and by inspection it can be seen that they add up to the original system.

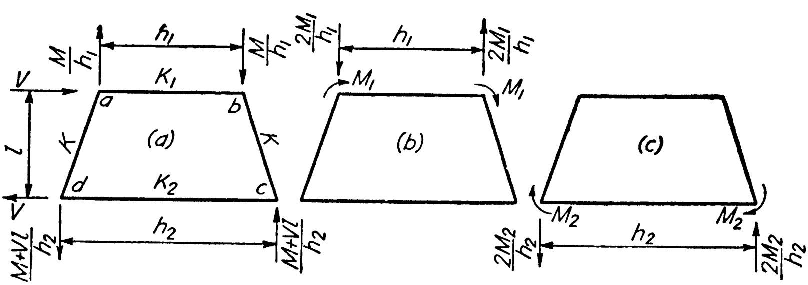


Fig. 69. Primary and secondary force systems acting on a panel.

36. Primary Moments. The force system shown in Fig. 69*a* represents the action of the external forces, and consequently the moments produced by these forces are necessary for *structural stability*. Therefore, as this force system represents the principal structural action,

the resulting moments will be called the *primary moments.* The values of the primary moments are given by the following equations:

$$
\begin{aligned}
M'_{ad} = M'_{bc} &= \frac{\alpha M - Vl}{2D}[3 + s + \alpha(2 + s)] \\
M'_{da} = M'_{cb} &= \frac{\alpha M - Vl}{2D}[3 + r + \alpha]
\end{aligned}
\tag{49}
$$

The equations for the primary moments are obtained by solving for the moments in Fig. 70*a* by means of the slope-deflection equations.

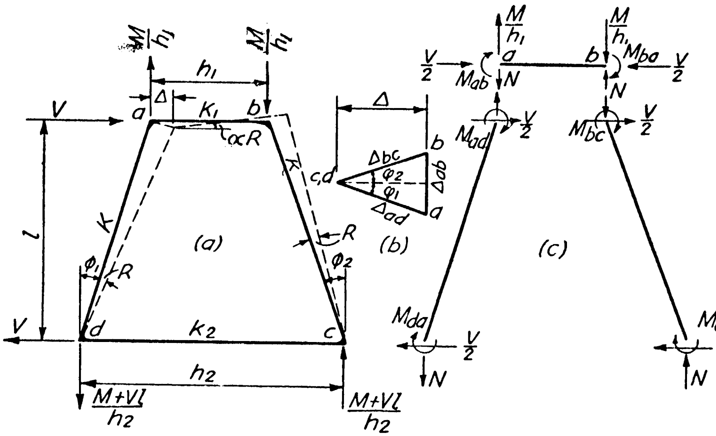


FIG. 70. Internal forces due to the primary force system.

From the displacement diagram in Fig. 70*b*, all translations can be expressed in terms of Δ, the horizontal movement of joints a and b. These values are

$$\frac{\Delta_{ad}}{ad} = \frac{\Delta}{\cos\phi_1}\cdot\frac{\cos\phi_1}{l} = \frac{\Delta}{l} = R \qquad \frac{\Delta_{bc}}{bc} = \frac{\Delta}{\cos\phi_2}\cdot\frac{\cos\phi_2}{l} = \frac{\Delta}{l} = R$$

$$\frac{\Delta_{ab}}{h_1} = \frac{\Delta(\tan\phi_1 + \tan\phi_2)}{h_1} = \frac{\Delta}{l}\cdot\frac{h_2 - h_1}{h_1} = \alpha R$$

It can be shown that $\theta_a = \theta_b$ and $\theta_d = \theta_c$ regardless of the values of the angles ϕ_1 and ϕ_2. Therefore the slope-deflection equations for the end moments will be (dropping the primes during the derivation)

$$
\begin{aligned}
M_{ab} &= M_{ba} = 6K_1\theta_a + 6K_1\alpha R \\
M_{ad} &= M_{bc} = 4K\theta_a + 2K\theta_d - 6KR \\
M_{da} &= M_{cb} = 2K\theta_a + 4K\theta_d - 6KR \\
M_{dc} &= M_{cd} = 6K_2\theta_d
\end{aligned}
$$

The equilibrium equations are

$$(a)\quad M_{ab} + M_{ad} = 0 \qquad (b)\quad M_{da} + M_{dc} = 0$$

and, from equilibrium of members ad and bc (Fig. 70c)($\Sigma M = 0$), $M_{ad} + M_{da} + M_{bc} + M_{cb} - N(h_2 - h_1) + Vl = 0$.

The equilibrium of member ab gives ($\Sigma M_b = 0$)

$$N = \frac{M_{ab} + M_{ba} + M}{h_1}$$

which make the third equilibrium equation equal to

$$(c)\quad M_{ad} + M_{da} + M_{bc} + M_{cb} - (M_{ab} + M_{ba} + M)\left(\frac{h_2 - h_1}{h_1}\right) + Vl = 0$$

By substituting the slope-deflection equations in the equilibrium equations (a), (b), and (c) and calling

$$\frac{K}{K_1} = r \qquad \frac{K}{K_2} = s \qquad \frac{h_2 - h_1}{h_1} = \alpha$$

we obtain

$$(a')\quad (2r + 3)\theta_a + r\theta_d - 3(r - \alpha)R = 0$$

$$(b')\quad s\theta_a + (2s + 3)\theta_d - 3sR = 0$$

$$(c')\quad (r - \alpha)\theta_a + r\theta_d - (2r + \alpha^2)R = \frac{\alpha M - Vl}{12K_1}$$

The solution of these equations gives the results

$$\begin{aligned}
\theta_a &= \frac{\alpha M - Vl}{12K_1}\left[\frac{-rs - 3r + 2\alpha s + 3\alpha}{rD}\right] \\
\theta_d &= \frac{\alpha M - Vl}{12K_1}\left[\frac{-rs - 3s - \alpha s}{rD}\right] \qquad (50) \\
R &= \frac{\alpha M - Vl}{12K_1}\left[\frac{-rs - 2r - 2s - 3}{rD}\right]
\end{aligned}$$

in which $D = 6 + r + s + \alpha(2\alpha + \alpha s + 2s + 6)$.

When the above values of θ_a, θ_d, and R are substituted back into the slope-deflection equations, the primary moments of equation 49 are obtained.

By means of equation 49, the primary moments M' are easily computed for each panel and recorded on a sketch of the structure. The

external moments M and Vl are considered positive when they act clockwise on the panel. The internal moments M' that act on the members ad and bc are also clockwise if positive. The value of α can be either positive or negative.

37. Secondary Moments. The force systems shown in Figs. 69*b* and *c* represent the effect of the internal moments in the adjacent panels and therefore exist as a result of *continuity* of the panels. As the moments M'' and M''' produced by these two force systems are not necessary for structural stability in the panel *abcd*, they will be defined as *secondary moments*. The magnitude of these secondary moments can be computed without difficulty by the following equations, which are derived in the same manner as equation 49.

$$M''_{ad} = M''_{bc} = +\frac{r}{D}M_1 \tag{51a}$$

$$M''_{da} = M''_{cb} = -\frac{r(1+\alpha)}{D}M_1 \tag{51b}$$

$$M'''_{ad} = M'''_{bc} = -\frac{s(1+\alpha)}{D}M_2 \tag{52a}$$

$$M'''_{da} = M'''_{cb} = +\frac{s(1+\alpha)^2}{D}M_2 \tag{52b}$$

Evidently the constants $\frac{r}{D}$, $\frac{r(1+\alpha)}{D}$, $\frac{s(1+\alpha)}{D}$, and $\frac{s(1+\alpha)^2}{D}$ are correction factors that can be computed for each panel and recorded on a sketch of the structure. The primary moments M' in the adjacent panels can be used for the first approximation of M_1 and M_2, and the corrections can then be computed as for any method of successive approximations. A convenient numerical arrangement for making the calculations is as follows:

(*a*) Compute and tabulate r, s, α, D, $\frac{r}{D}$, $\frac{r(1+\alpha)}{D}$, $\frac{s(1+\alpha)}{D}$, and $\frac{s(1+\alpha)^2}{D}$, for each panel.

(*b*) Compute the primary moments by equation 49, and record on a sketch of the frame.

(*c*) Make the necessary corrections by equations 51 and 52, using the recorded correction factors.

(*d*) Determine the sign of any secondary moment directly from the sign of the moment producing it by the following rule: Adjacent secondary moments have a sign opposite to that of the applied moment, while secondary moments at the far end of the panel have the same sign as the applied moment.

Example 24. The moments in the frame of Fig. 71 will be computed by the panel method. The calculation of the various constants that are recorded on the sketch requires no special explanation. The primary moments as given by equation 49 are

Panel 1

$$M'_{ab} = \frac{(-0.5)(0) - (5.45)(20)}{(2)(5.0)} [3 + 2.0 - (0.5)(2 + 2.0)] = -32.7$$

$$M'_{ba} = \frac{(-0.5)(0) - (5.45)(20)}{(2)(5.0)} [3 + 1.0 - 0.5] = -38.2$$

The moments and shears are taken from the left; therefore,

$$M = 0 \qquad V = 5.45 \qquad h_1 = 20 \qquad h_2 = 10$$

Panel 2 $\quad M'_{bc} = \dfrac{-(-4.55)(12)}{(2)(10.67)} [3 + 2.33] = +13.7$ Both ends

Panel 3 $\quad M'_{cd} = \dfrac{-(-4.55)(12)}{(2)(10.08)} [3 + 1.75] = +12.9$

$$M'_{dc} = \frac{-(-4.55)(12)}{(2)(10.08)} [3 + 2.33] = +14.5$$

The corrections are then made in the order indicated by the arrows (Fig. 71); that is, panel 2 was corrected for the moment -38.2 occurring in panel 1, the correction being $(0.219)(38.2)$ at either end. To determine the sign of the correction, the rule previously stated can be used. The correction at the far end of the panel has the same sign as, and that at the near end has a sign opposite to, the moment producing the correction. Thus, the corrections in panel 3 due to the moments $(+13.7 - 8.4) = +5.3$ acting on the left side will have a plus sign $(+1.2)$ on the far end and a minus sign (-1.2) at the adjacent end. The corrections in panel 1 due to $(+13.7 + 8.4 + 2.6) = +24.7$ will have a plus sign $(+4.9)$ at the far end and a minus sign (-2.5) at the adjacent end. After the corrections have been carried out to the de-

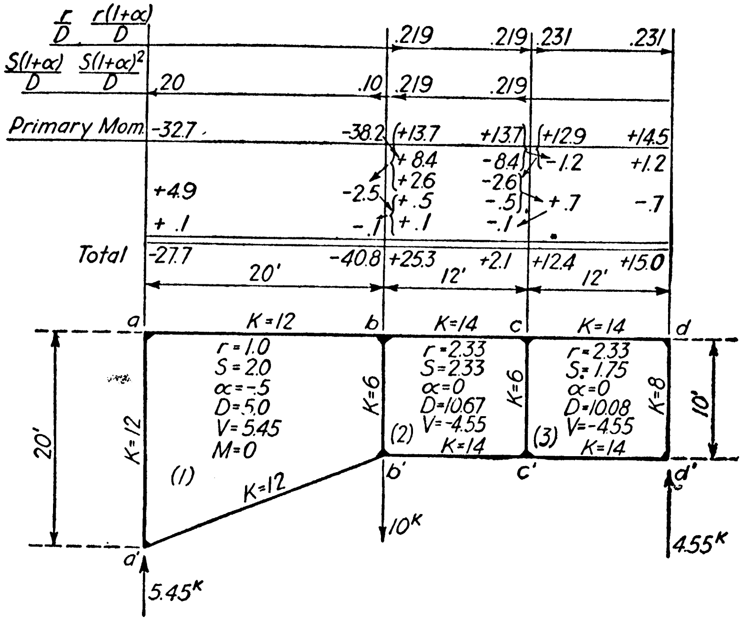


FIG. 71.

sired accuracy, the primary and secondary moments are added algebraically to give the actual value.

38. Frames Fixed at the Base. If the frame has columns that are fixed at the base, Fig. 72, the solution can be made by assuming a

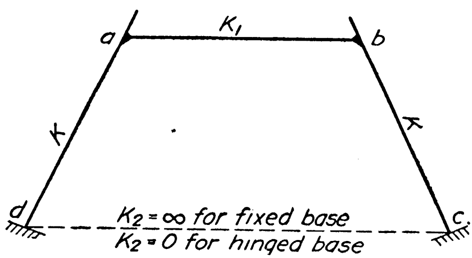


FIG. 72. Equivalent members for fixed and hinged bases.

member dc, shown by the dotted line, with $K_2 = \infty$. For this panel, since $s = \dfrac{K}{K_2} = 0$, the primary and secondary moments can be computed without difficulty.

39. Frames Hinged at the Base. If the frame is hinged at c and d (Fig. 72), an imaginary member dc with $K_2 = 0$ can be used. This

condition means that $s = \dfrac{K}{K_2} = \infty$. For this case the primary moments become:

$$M'_{ad} = M'_{bc} = \frac{\alpha M - Vl}{2} \lim_{s \to \infty} \frac{[(3+2)/s] + 1 + \alpha}{\dfrac{6 + r + 2\alpha^2 + 6\alpha}{s} + 1 + 2\alpha + \alpha^2}$$

$$= \frac{\alpha M - Vl}{2}\left[\frac{1}{1+\alpha}\right] \quad (53a)$$

$$M'_{da} = M'_{cb} = \frac{\alpha M - Vl}{2} \lim_{s \to \infty} \frac{(3 + r + \alpha)/s}{\dfrac{6 + r + 2\alpha^2 + 6\alpha}{s} + 1 + 2\alpha + \alpha^2}$$

$$= 0 \quad (53b)$$

The quantities in equation 53 were obtained by dividing the numerator and denominator of equation 49 by s. By dividing numerator and denominator of equation 51 by s, it can be easily seen that the secondary moment due to M_1 will be zero.

40. Moments in Triangular Panels ($\alpha = \infty$). In many Vierendeel trusses, it is desirable to use triangular end panels as shown in Fig. 73. For this case $h_1 = 0$ and $\alpha = \dfrac{h_2 - h_1}{h_1} = \infty$. The primary moments will then be

$$M'_{ac} = M'_{ad} = \lim_{\alpha \to \infty} \frac{\dfrac{M}{2}\left[\dfrac{3+s}{\alpha} + 2 + s\right] - \dfrac{Vl}{2}\left[\dfrac{3+s}{\alpha^2} + \dfrac{2+s}{\alpha}\right]}{\dfrac{6+r+s}{\alpha^2} + \dfrac{2s+6}{\alpha} + 2 + s} = \frac{M}{2} \quad (54a)$$

$$M'_{ca} = M'_{da} = \lim_{\alpha \to \infty} \frac{\dfrac{M}{2}\left[\dfrac{3+r}{\alpha} + 1\right] - \dfrac{Vl}{2}\left[\dfrac{3+r}{\alpha^2} + \dfrac{1}{\alpha}\right]}{\dfrac{6+r+s}{\alpha^2} + \dfrac{2s+6}{\alpha} + 2 + s} = \frac{M}{2}\left(\frac{1}{2+s}\right) \quad (54b)$$

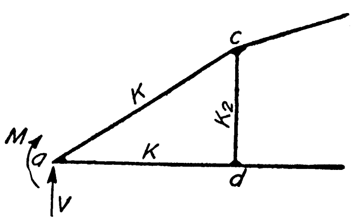


FIG. 73. Illustration of a triangular panel.

The values in equation 54 were obtained by dividing numerator and denominator of equation 49 by α^2.

The secondary moments will be

$$M'''_{ac} = M'''_{ad} = \lim_{\alpha \to \infty} \frac{s(1+\alpha)/\alpha^2}{D/\alpha^2} \mathrm{M}_2 = 0 \tag{55a}$$

$$M'''_{ca} = M'''_{da} = \lim_{\alpha \to \infty} \frac{s[(1/\alpha)+1]^2}{D/\alpha^2} M_2 = \left(\frac{s}{2+s}\right) M_2 \tag{55b}$$

Example 25. The end moments acting on all members of the bent in Fig. 61, Example 20, will be calculated by the panel method. In this structure the chord members (columns) of the panels are parallel; therefore all α values are zero.

The constants and primary moments for the various panels are

Panel 1 (1st story)

$$r = \tfrac{12}{14} = 0.86 \qquad s = \tfrac{12}{0} = \infty \qquad \alpha = 0$$

From Article 38, equations 53a and b,

$$M'_{ab} = M'_{a'b'} = 0 \qquad M'_{ba} = M'_{b'a'} = -\frac{Vl}{2} = -\frac{(20)(16)}{2}$$

$$= -160 \text{ ft-kips}$$

Panel 2 (2nd story)

$$r = \tfrac{10}{8} = 1.25 \qquad s = \tfrac{10}{14} = 0.714 \qquad D = 6 + r + s = 7.96$$

From equation 49, $\alpha = 0$

$$M'_{bc} = -\frac{Vl}{2D}(3+r) = -\frac{(12)(12)}{(2)(7.96)}(4.25) = -38.4$$

$$M'_{cb} = -\frac{Vl}{2D}(3+s) = -\frac{(12)(12)}{(2)(7.96)}(3.71) = -33.6$$

Panel 3 (3rd story)

$$r = \tfrac{6}{6} = 1.0 \qquad s = \tfrac{6}{8} = 0.75 \qquad D = 7.75$$

$$M'_{cd} = -\frac{(4)(12)}{(2)(7.75)}(3+1.0) = -12.4$$

$$M'_{dc} = -\frac{(4)(12)}{(2)(7.75)}(3+0.75) = -11.6$$

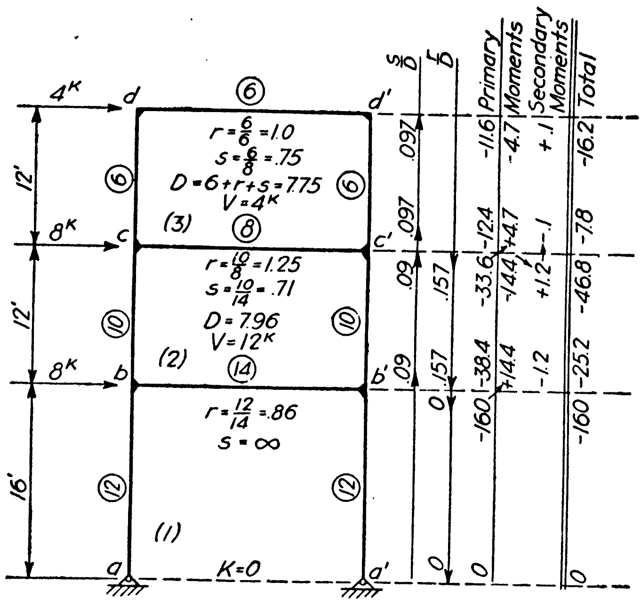


FIG. 74. Frame bent analyzed by the panel method.

The correction factors for the calculations of the secondary moments due to continuity of the panels are

Panel 1 $\quad \dfrac{r}{D} = \dfrac{r}{6 + r + s} = \dfrac{r/s}{[(6 + r)/s] + 1} = 0 \quad$ when $s = \infty$

therefore there is no correction in the first story.

$$\text{Panel 2} \qquad \frac{s}{D} = \frac{0.714}{7.96} = 0.09$$

$$\frac{r}{D} = \frac{1.25}{7.96} = 0.157$$

$$\text{Panel 3} \qquad \frac{s}{D} = \frac{0.75}{7.75} = 0.097$$

The primary moments and the correction factors are recorded on the diagram in Fig. 74, and the corrections are then made as indicated by the arrows. The moments in panel 2 that are caused by the end

moment -160 in panel 1 should be calculated first, as these are the largest corrections. These moments are

$$M''_{bc} = -(-160)\left(\frac{s}{D}\right) = (160)(0.09) = 14.4$$

$$M''_{cb} = (-160)\left(\frac{s}{D}\right) = -14.4$$

The correction at the opposite end of the panel is always the same sign as the moment acting on the panel; the adjacent correction is of opposite sign.

After these corrections are added to the primary moments in panel 2, panel 3 is corrected for the total effect, that is $-33.6 - 14.4$ or -48.0. These corrections are

$$M''_{cd} = -(-48.0)(0.097) = 4.7$$

$$M''_{dc} = -(48.0)(0.097) = -4.7$$

Panel 2 is then corrected for the total moment in panel 3, that is, for $-12.4 + 4.7$ or -7.7. These corrections are

$$M'''_{bc} = (-7.7)\left(\frac{r}{D}\right) = (-7.7)(0.157) = -1.2$$

$$M'''_{cb} = -(-7.7)(0.157) = +1.2$$

Panel 3 is then corrected for the additional moment $+1.2$ in panel 2, or

$$M''_{cd} = -(1.2)(0.097) = -0.12$$

$$M''_{dc} = (1.2)(0.097) = +0.12$$

Any additional corrections are too small to consider. The final moments check the values obtained in Example 20.

Example 26. The moments in the frame used in Example 22, Fig. 65*a*, will be calculated by the panel method. As the bent is hinged at the base, a member *af* with K equal to zero is assumed to be acting. The constants and primary moments for the panels are (Fig. 75)

Top panel

$$r = \tfrac{20}{100} = 0.2 \qquad s = \tfrac{20}{20} = 1.0 \qquad \alpha = \frac{15 - 12}{12} = 0.25$$

$$D = 6 + 0.2 + 1.0 + 0.25(0.5 + 0.25 + 2.0 + 6.0) = 9.4$$

$$V = 12 \text{ kips} \qquad M = 0$$

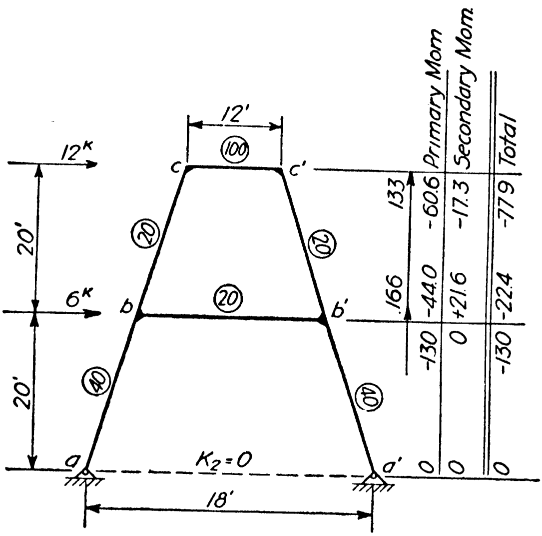


FIG. 75. Viaduct bent analyzed by the panel method.

From equation 49, the primary moments are

$$M'_{cb} = \frac{0 - (12)(20)}{(2)(9.4)}[3 + 1.0 + 0.25(2 + 1.0)]$$

or

$$M'_{cb} = (-12.76)(4.75) = -60.6 \text{ ft-kips}$$

and

$$M'_{bc} = (-12.76)(3 + r + \alpha) = (-12.76)(3.45) = -44.0 \text{ ft-kips}$$

Correction factors are (equation 52)

$$\frac{s(1+\alpha)}{D} = \frac{(1.0)(1.25)}{9.4} = 0.133 \qquad \frac{s(1+\alpha)^2}{D} = -(0.133)(1.25) = -0.166$$

Bottom panel

$$\alpha = \frac{18 - 15}{15} = 0.2 \quad M = (12)(20) = 240 \text{ ft-kips} \quad V = 18.0 \text{ kips}$$

From equation 53

$$M'_{ba} = \left[\frac{(0.2)(240) - (18)(20)}{2}\right]\frac{1}{1 + 0.2} = \frac{48 - 360}{(2)(1.2)} = -130$$

$$M'_{ab} = 0$$

Correction factors are zero for bottom panel.

The only corrections, as shown by the arrows in Fig. 75, are in the top panel, which must be corrected for the moment of -130 ft-kips in the bottom panel; that is,

$$M''_{cb} = (-130)(0.133) = -17.3$$

$$M''_{bc} = (-130)(-0.166) = +21.6$$

The actual moments in the top panel are therefore

$$M_{cb} = -60.6 - 17.3 = -77.9 \text{ ft-kips}$$

$$M_{bc} = -44.0 + 21.6 = -22.4 \text{ ft-kips}$$

In the bottom panel, the actual moments are equal to the primary moments, that is, zero and -130 ft-kips.

41. Effect of Panel Proportions. The term "panel proportions" refers here to the height of the verticals and the length of panels but does not include the stiffness of the various members. By means of equation 49, which neglects axial deformation, it can be seen that the primary moments, and therefore the total moments in any panel, will be zero if $\alpha M - Vl$ is equal to zero. This condition will be satisfied whenever the ratio $\alpha \left(\text{that is, } \dfrac{h_2 - h_1}{h_1}\right)$ is made equal to $\dfrac{Vl}{M}$ for the system of applied loads. If the applied load is uniformly distributed over the span, as it usually is for the dead weight, the equilibrium polygon is a parabola, and therefore a parabolic curve for the top or bottom chords will be most economical. The most uneconomical Vierendeel truss will be one with parallel chords. The variation in the end moments acting on the chords when the axis of the top chord is moved above or below a parabola can be readily seen from the influence diagrams in Fig. 76. For truss A, in which the joints of the top chord lie on a parabola, the positive and negative areas of the influence lines are equal, while for trusses B and C, which lie above and below a parabola, the areas are unequal.

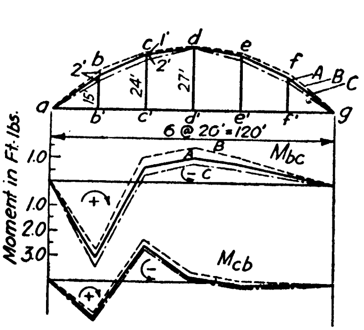


FIG. 76. Influence lines for moments in Vierendeel trusses.

The total combined dead- and live-load moments for the three different trusses of Fig. 76 are compared in Table 4. These moments were calculated from the influence diagrams for a dead load of 1500 lb per

linear foot of truss and for the standard H-20 truck loading plus 30 per cent impact. These values indicate that the maximum moments at certain sections for trusses *B* and *C* are 50 to 60 per cent higher than the corresponding moments for truss *A*. With a reduced dead load and a larger concentrated live load, truss *B* may give a more favorable comparison, but truss *C* is unsatisfactory for any practical loading.

The above comparison has been obtained on the assumption that the lengths of the members do not change. The error due to this assumption is relatively small except for shallow parallel-chord trusses with long panels. For parabolic-chord trusses the effect of axial stress on the magnitude of the end moments can be neglected unless the truss is unusually shallow. The moments caused by translation of the joints due to change in length of the members can be ascertained in the same manner as the so-called secondary stresses in triangular trusses. This problem is treated in the following section.

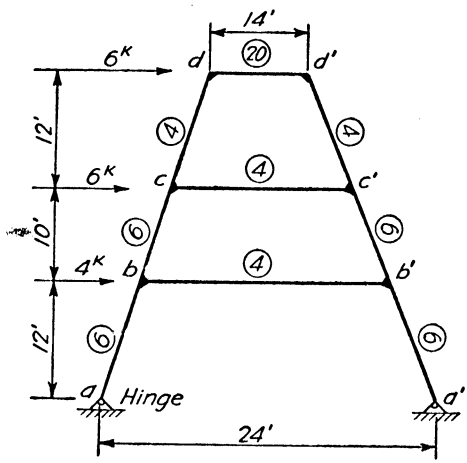
TABLE 4. SUMMARY OF MOMENTS FOR DEAD LOAD OF 1500 POUNDS PER FOOT OF TRUSS AND AMERICAN STANDARD H-20 HIGHWAY LOADING

Moment	Truss	Dead Load	Live Load + 30% Impact +	Live Load + 30% Impact −	Maximum Combined
	A	0	140.5	73.0	+140.5
M_{bc}	*B*	−48.7	125.0	97.8	−146.5
	C	+37.5	154.0	53.2	+191.5
	A	0	70.8	69.4	+70.8
M_{cb}	*B*	−25.3	66.1	82.3	−107.6
	C	+5.8	71.5	64.5	+77.3
	A	0	122.5	94.1	+122.5
M_{cd}	*B*	−17.6	113.8	101.6	−119.2
	C	+45.8	147.2	75.8	+193.0
	A	0	87.5	93.6	−93.6
M_{dc}	*B*	−35.0	72.5	115.3	−150.3
	C	+68.3	112.1	53.6	+180.4

The parabolic-chord Vierendeel truss is somewhat similar to the parabolic tied arch that is frequently constructed in the United States. The principal difference between these two types of structures is in the distribution of the bending moments. In the tied arch the bending moments are taken entirely by the arch rib, while in the Vierendeel truss they are distributed between the upper and lower chord members and the verticals. The total amount of material required for each structure is about the same.

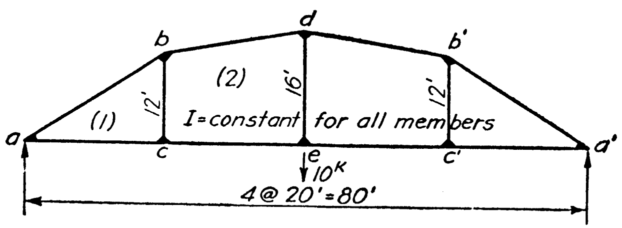
PROBLEMS

46. Solve Problems 41 and 42 by the panel method.
47. Compute the end moments for all members of the viaduct bent shown.



PROBLEM 47.

48. Calculate the end moments in all members of the Vierendeel truss shown by means of the panel method. (Note: Take advantage of symmetry by taking $K_{de} = \infty$, or, in other words, panel 2 can be assumed as fixed at d and e.)



PROBLEM 48.

Secondary Stresses in Triangular Trusses

42. Introduction. In the analysis of riveted or welded triangular trusses, the axial stresses are first computed on the assumption that the ends of the members are pin-connected. Such an assumption is likely to be far from the actual condition in shallow heavy trusses with large gusset plates. In such trusses, the joints are more comparable to those in a rigid frame in which end moments are produced by the continuity

that is established through welded or riveted connections. Any error caused by the assumption that the joints of a riveted or welded truss are rigid is partially offset by assuming that the moment of inertia of the members is constant to the center of the joint—an unnecessary assumption, however, as corrections for the increased moment of inertia within the joint can be approximated by methods explained in the next chapter.

The effect of secondary stresses upon the ultimate strength of triangular trusses has never been definitely determined. This subject is discussed in the various references at the end of the chapter, throughout which various conflicting opinions will be found. Undoubtedly the effect and therefore the importance of the secondary stresses will depend upon such factors as the manner of loading, type of connections, material used, and the proportions of the members.

43. Analytical Procedure. The following procedure is suggested for calculating the end moments and the corrections to the axial stresses that are caused by the use of rigid joints in triangular trusses:

(*a*) Calculate the axial stresses on the assumption that the ends of the members are pin-connected. The truss is usually loaded with full live and dead load although certain end moments may be a maximum under partial live load. If such variations are important, calculations can be made for a load at each panel point, although ordinarily such accuracy will not be necessary. The moments due to the dead load may be affected considerably by the method of erection.

(*b*) Draw a Williot displacement diagram, and scale the relative transverse displacement for each member. The rotation of the truss as a rigid body (Mohr rotation diagram) does not change its configuration and consequently does not affect the end moments.

(*c*) Calculate the fixed-end moments, $\dfrac{6EI\Delta}{L^2}$, for each member and the distribution factors at each joint.

(*d*) Distribute the fixed-end moments, starting at the joints with the greatest unbalance. Always carry over the correction to the next joint before balancing at that joint, as this procedure increases the rate of convergence. In applying the moment-distribution method, it is, of course, assumed that auxiliary forces are applied at each joint to prevent any additional translation due to flexure in the members.

(*e*) After the end moments have been determined in part (*d*), the shears in each member and the auxiliary forces at each joint should be calculated. If these auxiliary forces are large, they should be removed by applying equal and opposite forces and the corresponding axial stresses should be determined.

(*f*) Another Williot diagram can then be drawn for the corrections to the axial stresses that are obtained in part (*e*), and the corresponding end moments, shears, and axial stresses can be obtained as before. The final magnitudes of the end moments, shears, and axial stresses are therefore obtained as an alternating converging series which can be carried to any desired degree of accuracy.

Example 27. The application of the above method will be illustrated by calculating the secondary stresses for the truss in Example 2,

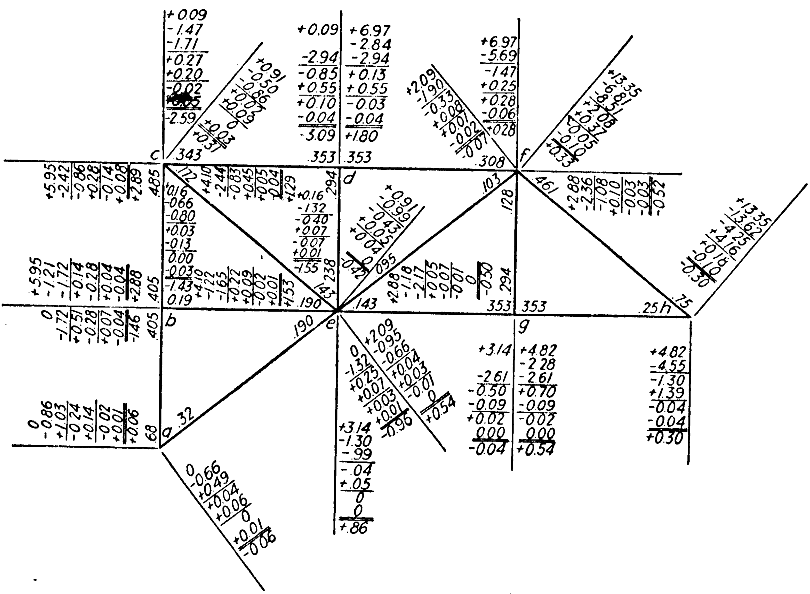


FIG. 77*a*. First approximation for end moments in a triangular truss.

Fig. 9*a*. The Williot diagram for this truss, shown in Fig. 26, requires no additional explanation. The relative displacements and the $\frac{I}{L}$ values as well as the fixed-end moments are tabulated in Table 5. The signs of the fixed-end moments were established directly by inspection of the displacement diagram. Distribution of the fixed-end moments is shown in Fig. 77*a*, in which the joints were balanced in the following order: *h*, *f*, *g*, *d*, *e*, *c*, *b*, *a*. The second and third cycles were made in the same order. It should be noted that the correction was carried over before the next joint was balanced, and by this procedure sufficient accuracy was obtained at the end of three cycles. The end moments and shears are large as the truss is short and heavy.

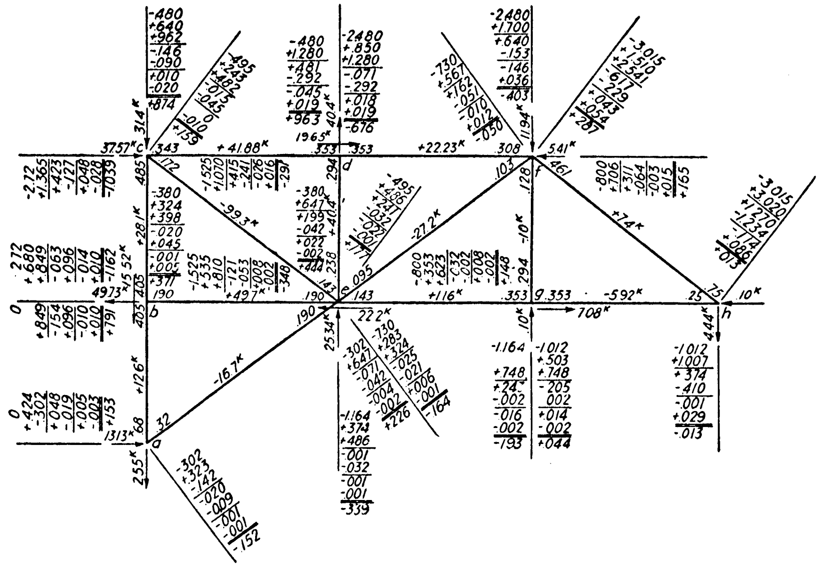


FIG. 77b. Second approximation for end moments in a triangular truss.

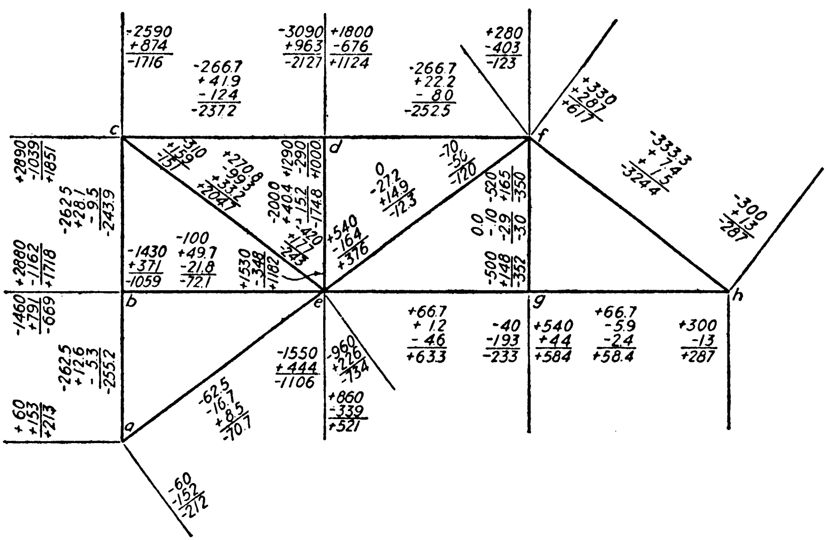


FIG. 77c. Summary of stresses in a triangular truss from successive approximations.

The shear and direct stress in each member was calculated from the first set of end moments, and by a summation of the horizontal and vertical components of these stresses at each joint the auxiliary forces recorded in Fig. 77*b* were obtained. The axial stress is also recorded on the diagram. Another Williot diagram (not shown) was drawn from the change in length of the members due to this axial stress, and another set of fixed-end moments was calculated. These fixed-end moments were distributed, and another set of axial stresses was calculated. The two sets of end moments and the three sets of axial stresses are recorded in Fig. 77*c*. It is evident that this procedure gives an alternating converging series for the true values.

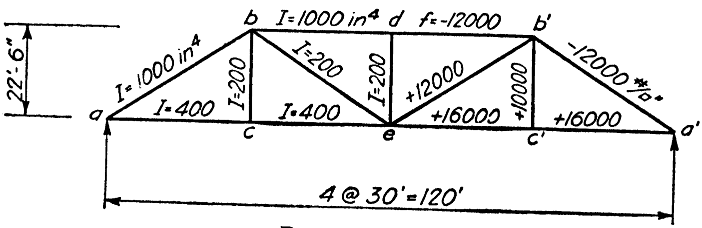
TABLE 5

$E = 29 \times 10^6$ lb per in.^{2}

Member	Length in.	$K = \frac{I}{L}$ in.^{3}	$\frac{6EI}{L^2} = \frac{174K}{L}$ (10^6)	Δ in.	Fixed-End Moment (10^6 in.-lb)
ab	144	17	20.55	0	0
ae	240	8	5.80	0	0
bc	144	17	20.55	+0.290	+5.95
be	192	8	7.24	+0.022	+0.16
cd	192	12	10.87	+0.008	+0.09
ce	240	6	4.35	+0.210	+0.91
de	144	10	12.10	+0.339	+4.10
df	192	12	10.87	+0.643	+6.97
ef	240	4	2.90	+0.720	+2.09
eg	192	6	5.43	+0.578	+3.14
fg	144	5	6.05	+0.477	+2.88
fh	240	18	13.05	+1.022	+13.35
gh	192	6	5.43	+0.888	+4.82

PROBLEMS

49. Compute the end moments in the Pratt truss shown. The moments of inertia and axial stress are recorded on the diagram of the truss. (Suggestion: Take advantage of symmetry by setting θ_d and θ_e equal to zero.)



PROBLEM 49.

50. Calculate the end moments in the Vierendeel truss of problem 48 due to the change in length of the chord members for vertical loads of 40 kips applied at points c, e, and c', if the area and moment of inertia of each member are 10.0 in.^{2} and 400 in.^{4} Neglect change in length of verticals.

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CHAPTER VI

CONTINUOUS GIRDERS AND FRAMES WITH VARIABLE MOMENT OF INERTIA

44. General Slope-Deflection Equations. The slope-deflection equations and methods of analysis that have been used in the preceding chapters for members with constant cross section also apply to members with variable cross section if the coefficients 4 and 2 are replaced by their proper values. For members with variable I, the slope-deflection equations take the general form:

$$
\begin{aligned}
M_{ab} &= \frac{EI_0}{L}\left[C_1\theta_a + C_2\theta_b \pm (C_1 + C_2)\frac{\Delta}{L}\right] \pm M_{Fab} \\
M_{ba} &= \frac{EI_0}{L}\left[C_2\theta_a + C_3\theta_b \pm (C_2 + C_3)\frac{\Delta}{L}\right] \pm M_{Fba}
\end{aligned}
\qquad (56)
$$

in which I_0 is the moment of inertia of some reference section. The value of the coefficients C_1, C_2, and C_3 and the fixed-end moments M_{Fab} and M_{Fba} depends upon the relative variations of I along the member with respect to I_0. The procedure for determining the magnitude of the coefficients and the fixed-end moments will be explained by numerical examples.

45. Members with One End Hinged. If one end of the member is hinged, for example at a, the moment M_{ab} is zero, and the moment M_{ba} can be expressed in terms of θ_b only. Thus, if

$$M_{ab} = K_0\left[C_1\theta_a + C_2\theta_b \pm (C_1 + C_2)\frac{\Delta}{L}\right] + M_{Fab} = 0$$

where

$$K_0 = \frac{EI_0}{L}$$

then

$$\theta_a = -\frac{M_{Fab}}{C_1K_0} - \frac{C_2}{C_1}\theta_b \pm \frac{C_1 + C_2}{C_1}\frac{\Delta}{L}$$

If this value of θ_a is substituted in the equation for M_{ba}, the simplified form of the equation is

$$M_{ba} = K_0\left[\left(C_3 - \frac{C_2^2}{C_1}\right)\left(\theta_b \pm \frac{\Delta}{L}\right)\right] + M_{Fba} - \frac{C_2}{C_1} M_{Fab} \quad (57a)$$

If the beam is hinged at b, then the simplified form of the equation for M_{ab} will be:

$$M_{ab} = K_0\left[\left(C_1 - \frac{C_2^2}{C_3}\right)\left(\theta_a \pm \frac{\Delta}{L}\right)\right] + M_{Fab} - \frac{C_2}{C_3} M_{Fba} \quad (57b)$$

46. Calculation of C_1, C_2, and C_3. If the fixed-end moments are temporarily omitted, the first problem is to express the relation between the end moments applied to any member ab and the angular rotation of the end tangents θ_a and θ_b. This relation is easily determined in the following manner: First, calculate the angles β_{a1} and β_{b1}, Fig. 78a, due to a unit moment applied at a. Then calculate the

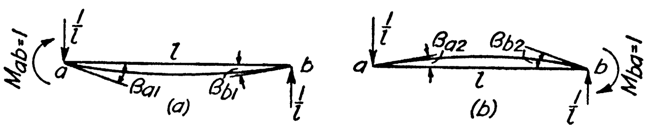


FIG. 78. Rotation of end tangents due to unit end moments.

angles β_{a2} and β_{b2}, Fig. 78b, due to a unit moment applied at b. By the reciprocal theorem, we know that β_{b1} is equal to β_{a2}. For determining the numerical values of the angles, the conjugate-beam method is convenient.

For any end moments M_{ab} and M_{ba} applied simultaneously, the rotation of the end tangents θ_a and θ_b must be:

$$\begin{aligned} \theta_a &= M_{ab}\beta_{a1} - M_{ba}\beta_{a2} \\ \theta_b &= -M_{ab}\beta_{b1} + M_{ba}\beta_{b2} \end{aligned} \quad (58a)$$

Solving these equations for M_{ab} and M_{ba} and making β_{a2} equal to β_{b1}, we obtain

$$\begin{aligned} M_{ab} &= \frac{\beta_{b2}}{A}\theta_a + \frac{\beta_{b1}}{A}\theta_b \\ M_{ba} &= \frac{\beta_{b1}}{A}\theta_a + \frac{\beta_{a1}}{A}\theta_b \end{aligned} \quad (58b)$$

in which

$$A = \beta_{a1}\beta_{b2} - \beta_{b1}^2$$

These equations show that the coefficient of θ_b in the equation for M_{ab} will be identical with the coefficient for θ_a in the equation for M_{ba}.

If $\frac{EI_0}{L}$ is represented by K_0, the value of the coefficients will be

$$C_1K_0 = \frac{\beta_{b2}}{A} \qquad C_2K_0 = \frac{\beta_{b1}}{A} \qquad C_3K_0 = \frac{\beta_{a1}}{A} \tag{58c}$$

The absolute values of β must be used in the above equations as the correct signs have already been considered in deriving the expressions.

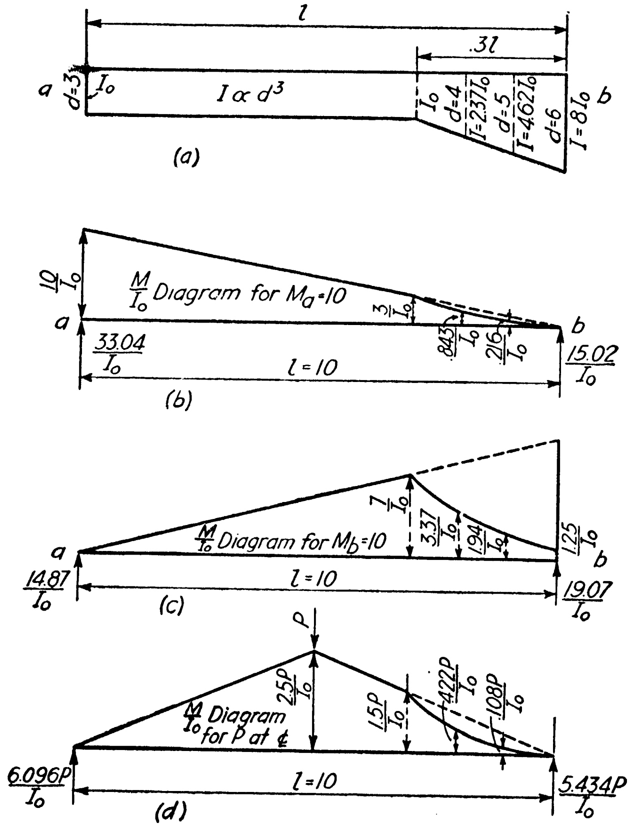


FIG. 79. M/I diagrams for a beam with variable I.

47. Calculation of M_{Fab} and M_{Fba}. The value of the fixed-end moments for any applied loads can be readily computed after the coefficients C_1, C_2, and C_3 have been determined. The most direct procedure is to compute the angular rotations α_a and α_b of the end tangents

due to the applied loads when the member is considered simply supported. Since the fixed-end moments must prevent any rotation of the end tangents, they must satisfy the relationship

$$\begin{aligned} M_{Fab} &= C_1K_0(-\alpha_a) + C_2K_0(-\alpha_b) \\ M_{Fba} &= C_2K_0(-\alpha_a) + C_3K_0(-\alpha_b) \end{aligned} \tag{59}$$

In all the above discussion, clockwise rotation is considered positive.

Example 28. The coefficients C_1, C_2, and C_3, for the member ab, Fig. 79a, will be computed by the above method. The angles are conveniently determined by the conjugate-beam method; that is, the values of β are numerically equal to the reactions of a simply supported beam loaded with the proper $\dfrac{M}{EI}$ diagram. To determine β_{a1} and β_{b1} an end moment of 10 units is applied at end a, and the length L is divided into 10 units. The $\dfrac{M}{I_0}$ diagram is illustrated in Fig. 79b, I_0 being taken as the I of the constant portion. The reactions for this $\dfrac{M}{I}$ diagram considered as an applied weight are: $\dfrac{33.04}{I_0}$ at a and $\dfrac{15.02}{I_0}$ at b. In computing the reactions due to the $\dfrac{M}{I}$ diagram, the numerical work is simplified if the curved portions are divided into triangles with constant altitude as shown in Fig. 80. The area of any two triangles such

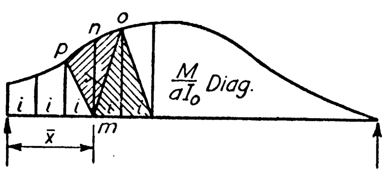


FIG. 80.

as $mnop$ is $\dfrac{(2)(mn)i}{2}$, or yi, and the center of gravity lies on mn. The angles β_{a1} and β_{b1} (E times the actual angles) for a unit moment at a and for any span L are

$$\beta_{a1} = 0.3304\frac{L}{I_0} \qquad \text{and} \qquad \beta_{b1} = 0.1502\frac{L}{I_0}$$

In the same manner the values of β_{a2} and β_{b2} are determined by apply-

ing an end moment of 10 units at b. The $\frac{M}{I_0}$ diagram for this moment is given in Fig. 79c, and the values of the reactions of the conjugate beam are $\frac{14.87}{I_0}$ at a and $\frac{19.07}{I_0}$ at b. The end rotations are, therefore,

$$\beta_{a2} = \frac{0.1487L}{I_0} \qquad \beta_{b2} = \frac{0.1907L}{I_0}.$$

However, we have already stated that β_{b1} must equal β_{a2}; consequently the difference in the above calculations is indicative of the numerical accuracy of the computations. In this problem, it is sufficiently accurate to use an average value of β_{b1} equal to $\frac{0.1495\,L}{I_0}$. When the above values of β's are substituted in equation 58b, the following equations (with value of E replaced) are obtained:

$$A = \beta_{a1}\beta_{b2} - \beta_{b1}^2 = 0.063 - 0.0224 = 0.0406$$

$$M_{ab} = \frac{EI_0}{L}\left(\frac{0.1907}{0.0406}\theta_a + \frac{0.1495}{0.0406}\theta_b\right)$$

or

$$M_{ab} = \frac{EI_0}{L}(4.68\theta_a + 3.68\theta_b)$$

$$M_{ba} = \frac{EI_0}{L}\left(\frac{0.1495}{0.0406}\theta_a + \frac{0.3304}{0.0406}\theta_b\right)$$

$$M_{ba} = \frac{EI_0}{L}(3.68\theta_a + 8.13\theta_b)$$

The fixed-end moments for a concentrated load P at the center of the span will also be determined. The $\frac{M}{I_0}$ diagram for the beam when considered simply supported is given in Fig. 79d. The reactions for this conjugate beam are $\frac{6.096P}{I_0}$ at a, and $\frac{5.434P}{I_0}$ at b. The angular rotations for any span L are, therefore,

$$\alpha_a = \frac{0.06096PL^2}{EI_0} \qquad \alpha_b = -\frac{0.05434PL^2}{EI_0}$$

When these angles are substituted in equation 59, the fixed-end moments are

$$M_{Fab} = \frac{EI_0}{L}\left[(4.68)\left(-0.06096\frac{PL^2}{EI_0}\right) + (3.68)\left(0.05434\frac{PL^2}{EI_0}\right)\right]$$
$$= -0.088PL$$

$$M_{Fba} = \frac{EI_0}{L}\left[(3.68)\left(\frac{-0.06096PL^2}{EI_0}\right) + (8.13)\left(\frac{0.05434PL^2}{EI_0}\right)\right]$$
$$= 0.218PL$$

If a relative displacement Δ is also considered, the complete equations become

$$M_{ab} = \frac{EI_0}{L}\left[4.68\theta_a + 3.68\theta_b + 8.36\frac{\Delta}{L}\right] - 0.088PL$$

$$M_{ba} = \frac{EI_0}{L}\left[3.68\theta_a + 8.13\theta_b + 11.81\frac{\Delta}{L}\right] + 0.218PL$$

If the beam ab in Fig. 79 is hinged at a, the moment at b is

$$M_{ba} = \frac{EI_0}{L}\left(8.13 - \frac{3.68^2}{4.68}\right)\left(\theta_b + \frac{\Delta}{L}\right) + 0.218P + \left(\frac{3.68}{4.68}\right)(0.088PL)$$
$$= \frac{EI_0}{L}\left[5.23\left(\theta_b + \frac{\Delta}{L}\right)\right] + 0.287PL$$

48. Charts for Coefficients C_1, C_2, and C_3. The procedure described for computing the values of the coefficients C_1, C_2, and C_3 in terms of some reference section whose moment of inertia is I_0 is by no means difficult but is usually time-consuming if many members are involved. For this reason tables and charts that give the values of the coefficients and fixed-end moments for particular types of members are often constructed (see Refs. 6·1 and 6·3). In general these charts are for members which have rectangular cross sections with constant width and with depths that vary according to some definite shape, such as: (*a*) beams with straight or parabolic haunches at one or both ends; (*b*) beams with infinite cross section at one or both ends.

Members with other types of cross section can frequently be solved by converting them into members with equivalent rectangular sections, that is, cross sections with the same variation in I. This can be done by calculating the value of I for the actual member at several sections and then determining the equivalent depth d for the substi-

tute member from the relation that $\frac{d}{d_0}$ is equal to $\sqrt[3]{\frac{I}{I_0}}$, where d_0 and I_0 refer to the reference section. Approximate values of the coefficients can then be obtained from the charts for rectangular sections. The use of the diagrams in the Appendix will be illustrated later by numerical examples.

49. Effect of Construction Details upon C_1, C_2, and C_3. The preceding discussion of the determination of the coefficients C_1, C_2, and C_3 does not consider the practical problem of how the deformation of material within the joint, or, rather, inside a zone that is more or less common to several intersecting members, affects their values. A more comprehensive discussion of this subject as related to practical design will be presented later, but at the present time the following assumptions seem warranted by the limited experimental results that are available.

(*a*) *Rectangular Cross Sections.* When structural members with rectangular cross section intersect with sharp corners, Fig. 81*a*, the effect

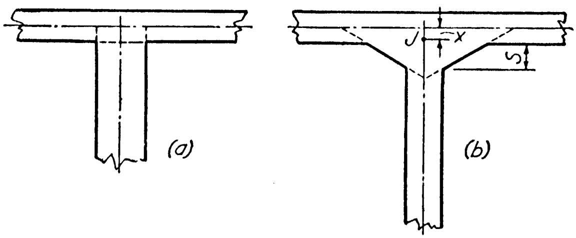


FIG. 81. Effective cross section within the joint.

of the deformation of the member within the joint can be obtained with reasonable accuracy by assuming a constant cross section to the center lines of the members. The omission of the deformations due to shear and stress concentration more than compensates for any increase in depth within the joint. For the calculation of bending moments, the assumption of constant cross section is therefore sufficiently accurate. For tapered members with sharp corners, Fig. 81*b*, it is sufficiently accurate to continue the inclined edge to the center line of the members. For members with curved haunch, Fig. 82, it is recommended that the material within the joint, shown by the shaded area, be assumed as having no deformation or that I equals infinity. In the curved portion the value of I is computed for the transverse section x-x by assuming that the full cross section is effective although the stress distribution is by no means linear. This variation from a linear stress distribution is particularly important when flange sections are used.

The theoretical center of the joint is at some point, J, slightly below the intersection of the center lines as shown in Figs. 81 and 82. Tests made by the author on celluloid models showed that the value of x is not more than $\frac{S}{8}$ for straight 45° haunches or $\frac{R}{8}$ for circular ones. The

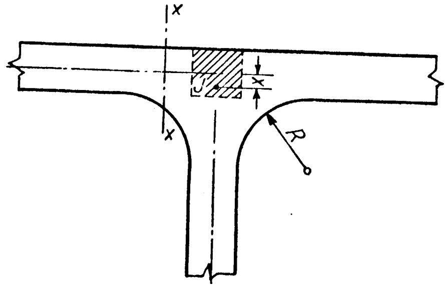


FIG. 82. Joints with curved fillets.

intersection of the bottom of the beam with the center line of the column is recommended as a working point for relatively deep haunches.

(b) *I-Shaped Sections.* Both theoretical and experimental data prove that the distribution of normal stress across sections in members with variable depth when subjected to flexure is a non-linear one (see Ref. 6·8). In a member with a rectangular cross section of constantly varying depth which is subjected to pure flexure, the normal and shearing stresses are distributed across the section as indicated in Fig. 83a. An exact solution of this problem shows that the normal

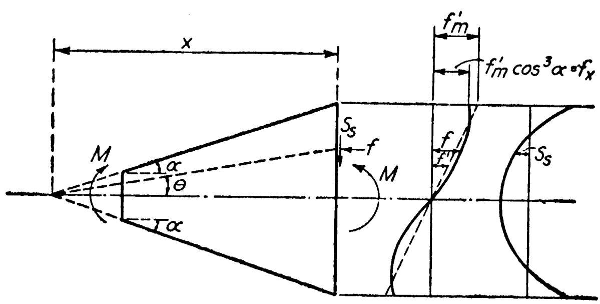


FIG. 83a. Stress distribution in tapered members for pure bending.

stress f_x on the outer fibers is practically equal to $f' \cos^3 \alpha$, in which f' is the stress computed by the usual flexure formula on the basis of a linear distribution of stress. If flanges are added, the distribution across the web is not materially altered, and the average stress in the flange will not be more and is probably less than at the edge of the web.

Such a distribution would give, on a section normal to the axis of an I-member, such as section 1–1, Fig. 83*b*, a total flange component parallel to the axis of N equal to $f_x \dfrac{A_f}{\cos\alpha}$, where α is the angle of inclination of the flange to the axis. Since f_x is equal to $f' \cos^3 \alpha$, this total flange component N equals $A_f\, f' \cos^2 \alpha$, and, therefore, if a straight-line distribution of stress is assumed, an equivalent flange area of $A_f \cos^2 \alpha$ should be used to determine the effective cross section.

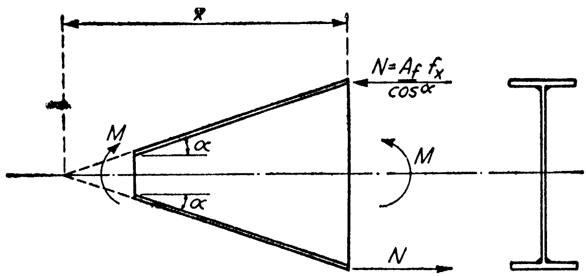


FIG. 83*b*. Resultant normal flange stress in a tapered I member for pure bending.

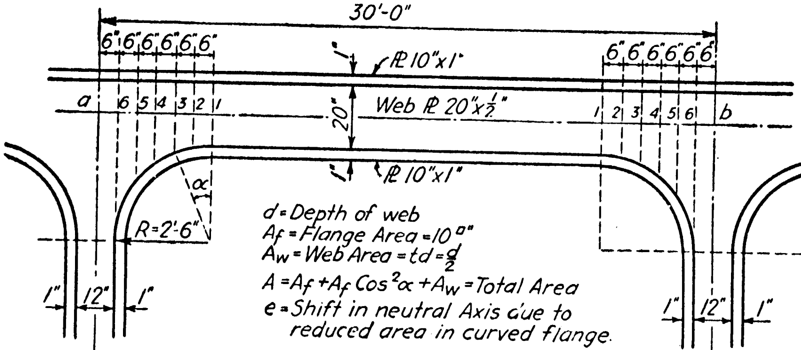
The web area can be used without reduction. The principal stress p in the flange parallel to the boundary is

$$p = \frac{f_x}{\cos^2 \alpha} = f' \cos \alpha$$

In computing the coefficients C_1, C_2, and C_3 of the slope-deflection equations, the reduced flange area $A_f \cos^2 \alpha$ should be substituted for the actual area. If curved flange sections without radial stiffeners are used, there will be an additional reduction of the effective flange area because of the variation in stress due to inward radial bending. This effect can be neglected in computing the coefficients, although it is important in stress and stability calculations.

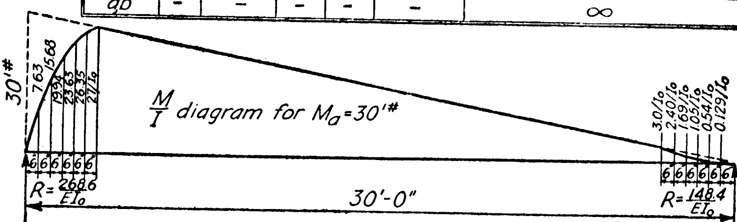
Although the equivalent flange area $A_f \cos^2 \alpha$ was determined from the actual stress distribution in a straight tapered section under pure bending, experimental data indicate that it is also a good approximation for curved flange sections. The normal stress on sections perpendicular to the axis of the straight portion of the member, which is considered here, should not be confused with the normal stress on radial sections. This latter distribution is entirely different as it increases sharply toward the boundary. If concentrated loads are applied in the vicinity of curved flanges, shear and direct compression stresses may alter considerably the above distribution for pure bending.

Example 29. The coefficients C_1, C_2, and C_3 will be computed for the member ab in Fig. 84. The reduced flange area $A_f \cos^2 \alpha$ is used for sections through the curved fillet. The only difference between this



$$e = \frac{A_f(1-\cos^2\alpha)}{A}\frac{(d+1)}{2} \qquad I = \frac{d^3}{24} + \frac{(10+10\cos^2\alpha)(d+1)^2}{4} - Ae^2$$

Section	α	$A_f \cos^2\alpha$	d	A	e	I
1	0	10.0	20.0	30.0	0	333 + 2205 - 0 = 2538 = I_o
2	11°33'	9.6	20.6	29.9	0.15	364 + 2285 - 0 = 2649 = 1.044I_o
3	23°35'	8.4	22.5	29.7	0.63	475 + 2540 - 12 = 3003 = 1.182I_o
4	36°52'	6.4	26.0	29.4	1.65	732 + 2980 - 80 = 3632 = 1.43I_o
5	53°08'	3.6	32.0	29.6	3.56	1365 + 3700 - 375 = 4690 = 1.85I_o
6	90°00'	0	50.0	35.0	7.30	5200 + 6500 - 1870 = 9830 = 3.87I_o
ab	-	-	-	-	-	∞



$$\beta_{a_1} = \beta_{b_2} = \frac{(268.6)}{(30)(30)}\frac{1}{EI_o} = \frac{0.2984}{K_o} \qquad \beta_{b_1} = \beta_{a_2} = \frac{148.4}{(30)(30)}\frac{1}{EI_o} = \frac{0.165}{K_o}$$

$$A = (0.2984)^2 - (0.165)^2 = 0.0618$$

$$C_1 = C_3 = \frac{0.2984}{0.0618} = 4.82$$

$$C_2 = \frac{0.165}{0.0618} \doteq 2.67$$

FIG. 84. Calculation of coefficients for slope-deflection equations for members with curved fillets.

problem and Example 28 is in the calculation of the moment of inertia for the various sections. The recommended procedure for computing I is illustrated in Fig. 84. The angle α and depth d are computed first for each section. The upward shift e of the neutral axis from the center

of the web due to the reduction of effective area in the curved bottom flange is then computed. The moment of inertia about the shifted axis will be equal to the I about the center axis minus Ae^2, or

$$I = \frac{td^3}{12} + \frac{A_f + A_f \cos^2 \alpha}{4} (d + 1)^2 - Ae^2$$

The calculations for I are summarized in Fig. 84 together with the determination of the β values as in Example 28. The $\frac{M}{I}$ diagram for a moment M_a equal to 30 is drawn with the ordinates expressed in terms of I_0. The reactions due to the $\frac{M}{I}$ diagram, therefore, give the angular rotations of the end tangents. The rotations β for a unit end moment and any span L are obtained by multiplying the reactions by $\frac{L}{900}$. The rotations for a unit end moment at b can be obtained from the same diagram because of symmetry. The coefficients C_1, C_2, and C_3 are computed from equations 58c as in Example 28.

50. Distribution and Carry-over Factors. Once the coefficients of the slope-deflection equations are known for all members of a frame, the distribution and carry-over factors can be determined in the same manner as for members with constant moment of inertia. The distribution factor at any joint then becomes

$$r = \frac{CK_0\theta_a}{\Sigma CK_0\theta_a} = \frac{CK_0}{\Sigma CK_0} \tag{60}$$

and the carry-over factors are $\frac{C_2}{C_1}$ and $\frac{C_2}{C_3}$; that is, from equation 56

$$M_{ba} = \frac{C_2}{C_1} M_{ab} \quad \text{when} \quad \theta_b = 0$$

and

$$M_{ab} = \frac{C_2}{C_3} M_{ba} \quad \text{when} \quad \theta_a = 0$$

The principal source of error is in interchanging the coefficients C_1 and C_3 when a member is unsymmetrical. If there is any doubt, the slope-deflection equations should be written out and the distribution ratio of equation 60 used with the angle θ in it. The application of the moment-distribution method to members with variable cross section will be illustrated by a numerical example.

Example 30. The end moments acting on the various members of the frame in Fig. 85 will be computed by the moment-distribution method. The coefficients and fixed-end moments for members *ab* and *bc* can be obtained from the diagrams on pages 321 to 327 in the

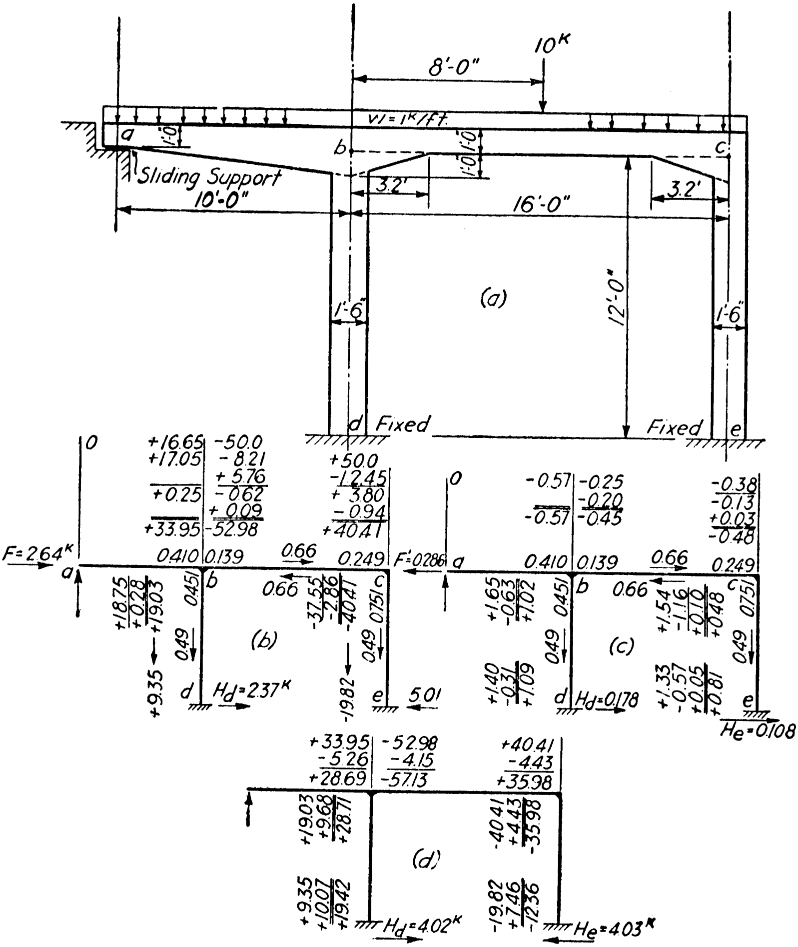


FIG. 85. Analysis of a continuous frame with members having variable *I*.

Appendix, which were prepared by the Portland Cement Association. In these diagrams the coefficients C_1 and C_3 are designated by *k*, the stiffness coefficient, and the carry-over factors $\frac{C_2}{C_1}$ and $\frac{C_2}{C_3}$ by *C*. The coefficients for the columns *bd* and *ce* will be calculated by the method used in Example 28.

Member ab. If the member is first assumed to be restrained at both ends, the coefficient $C_1 = k$ from page 324, diagram 1 ($a = 1.0$, $\frac{\min d}{\max d} = 0.5$), is 6.9 and the carry-over factor $C = \frac{C_2}{C_1} = 0.84$ from diagram 2. The value of C_2 is therefore $(0.84)(6.9) = 5.80$. From diagram 3, the fixed-end moment at a is

$$M_{Fab} = -0.053wL^2 = -(0.053)(1)(10)^2 = -5.3 \text{ ft-kips}$$

From page 323, for the haunched end, $k = C_3 = 19.4$ and $C = \frac{C_2}{C_3} = 0.3$. Therefore, $C_2 = (0.3)(19.4) = 5.82$.

The fixed-end moment at b is

$$M_{Fba} = 0.122wL^2 = (0.122)(1)(10)^2 = +12.2 \text{ ft-kips}$$

As the end a is free to rotate, equation 57*a* should be used, that is,

$$M_{ba} = \left(19.4 - \frac{5.8^2}{6.9}\right) K_0 \left(\theta_b + \frac{\Delta}{L}\right) + 12.2 - (0.84)(-5.3)$$

or

$$M_{ba} = 14.5K_0 \left(\theta_b + \frac{\Delta}{L}\right) + 16.65$$

The value of K_0 for a rectangular cross section with a width of one foot is

$$K_0 = \frac{I_0}{L} = \frac{(1.0)^4}{(12)(10)} = 0.00833$$

Member bc. From page 321, diagram 1, for $a = 0.2$, $\frac{\min\ d}{\max\ d} = 0.5$, the coefficients are $k = C_1 = C_3 = 7.8$.

From diagram 2

$$C = \frac{C_2}{C_1} = \frac{C_2}{C_3} = 0.66 \qquad \therefore C_2 = (0.66)(7.8) = 5.15$$

The fixed-end moments are (diagrams 3 and 4)

$$M_{Fbc} = -[(0.154)(10)(16) + (0.0992)(1)(16)^2] = -50.0 \text{ ft-kips}$$

$$M_{Fcb} = +50.0 \text{ ft-kips}$$

Member bd. The coefficients for member *bd* will be computed on the assumption that I equals ∞ for a distance of 1.0 ft at the top of the column. For a moment of 12 ft-kips applied at end d for the column *bd*, the $\frac{M}{I}$ values are shown in the diagram in Fig. 86*a*. The $\frac{M}{I}$ diagram

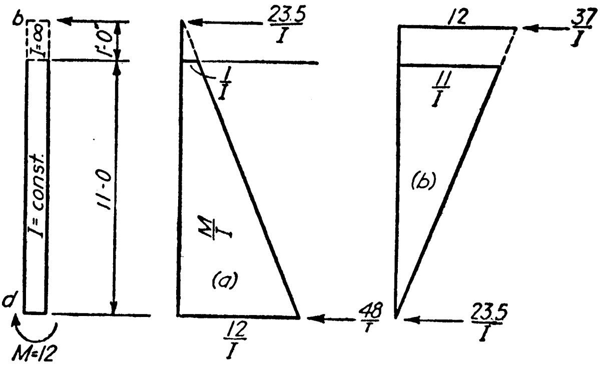


FIG. 86.

for a moment of 12 ft-kips at b is shown in Fig. 86b. The reactions of the conjugate beam when loaded with these diagrams determine the end rotations. For a unit end moment and any span L, the end rotations must be multiplied by $\frac{L}{144}$ or,

$$\beta_{d1} = \frac{48.0L}{144EI} = \frac{L}{3EI}$$

$$\beta_{b1} = \beta_{d2} = \frac{23.5L}{144EI} = 0.163\frac{L}{EI}$$

$$\beta_{b2} = \frac{37.0L}{144EI} = 0.256\frac{L}{EI}$$

$$A = (0.333)(0.256) - (0.163)^2 = 0.0586$$

$$C_1 = \frac{\beta_{b2}}{A} = \frac{0.256}{0.0586} = 4.37$$

$$C_2 = \frac{\beta_{b1}}{A} = \frac{0.163}{0.0586} = 2.78$$

$$C_3 = \frac{\beta_{d1}}{A} = \frac{0.333}{0.0586} = 5.68$$

Member ce. The coefficients for member ce will be determined in the same manner as for bd, except that $I = \infty$ will be taken for a distance of 0.8 ft instead of 1.0 ft. The values of the coefficients are

$$C_1 = 4.2 \qquad C_2 = 2.6 \qquad C_3 = 5.3$$

Distribution and Carry-over Factors

Joint b

Member	C	K	CK	$r = \frac{CK}{\Sigma CK}$	Carry-over
ba	14.5	0.00833	0.121	0.410	0
bc	7.8	$\frac{(1.0)^3}{(12)(16)} = 0.0052$	0.041	0.139	0.66
bd	5.68	$\frac{(1.5)^3}{(12)(12)} = 0.0234$	0.133	0.451	0.49
			$\Sigma CK = 0.295$	$\Sigma r = 1.000$	

Joint c

Member	C	K	CK	$r = \frac{CK}{\Sigma CK}$	Carry-over
cb	7.8	0.0052	0.041	0.249	0.66
ce	5.3	0.0234	0.124	0.751	0.49
			$\Sigma CK = 0.165$	$\Sigma r = 1.000$	

After the fixed-end moments, distribution factors, and carry-over factors are determined, the solution is made in the same manner as for members with constant cross section. An auxiliary force, F, is applied to prevent sidesway while the fixed-end moments are corrected for rotation of the joints. This part of the solution is presented in Fig. 85*b*. The value of F must equal the difference between H_d and H_e or 2.64 kips.

To remove the auxiliary force F, the frame will be given a horizontal movement of 100 units to the left, for which

$$M_{Fbd} = (C_2 + C_3)K_0 \frac{\Delta}{L} = (2.78 + 5.68)(0.0234)\left(\frac{100}{12}\right) = 1.65$$

$$M_{Fdb} = (C_1 + C_2)K_0 \frac{\Delta}{L} = (4.37 + 2.78)(0.0234)\left(\frac{100}{12}\right) = 1.40$$

$$M_{Fce} = (5.3 + 2.6)(0.0234)\left(\frac{100}{12}\right) = 1.54$$

$$M_{Fec} = (4.2 + 2.6)(0.0234)\left(\frac{100}{12}\right) = 1.33$$

The distribution of these fixed-end moments is shown in Fig. 85*c*.

The force F' that is necessary to give these moments is found to be

$$F' = 0.178 + 0.108 = 0.286 \text{ kips}$$

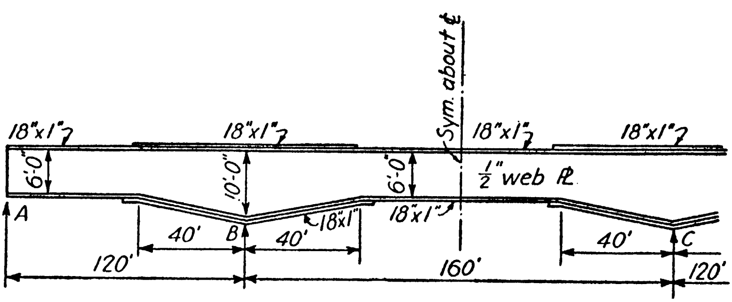
Therefore, the force F can be removed by multiplying the moments in Fig. 85c by $\frac{2.64}{0.286}$ and adding the resulting moments to the corresponding values in Fig. 85b. The final moments are shown in Fig. 85d.

PROBLEMS

51. (a) Calculate the coefficients C_1, C_2, and C_3 for member ab of Fig. 85a, and compare with the values given by the diagrams.

(b) Calculate the fixed-end moments for a uniform load of 1 kip per foot, and compare with the values in the diagrams.

52. (a) Calculate the moments over the supports B and C for the continuous girder shown for a uniform load of 2000 lb per linear foot on the entire structure.



PROBLEM 52.

Compare these moments with the values that would be obtained for a similar girder with constant cross section. Neglect the reduction in the effective area of the compression flange due to its inclination.

(b) Compare the moment at the center of the structure with the corresponding value for a girder of constant cross section.

53 (a) Design the frame shown for the following loads:

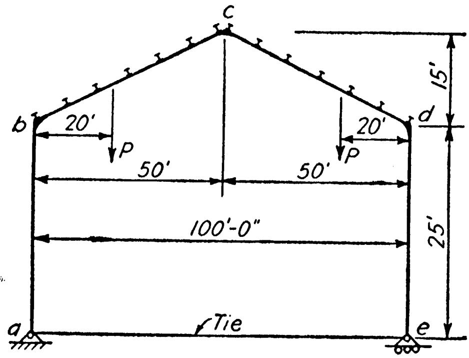
Dead load of roof	15 lb per ft^2
Estimated dead load of frame and bracing	5 lb per ft^2
Snow load	20 lb per ft^2
Wind load	20 lb per ft^2 on a vertical surface

Crane load P = 10,000 lb

Spacing of frames is 20 ft 0 in. center to center.

Use rolled steel sections with increased depth at b, c, d. A.I.S.C. specifications.

(b) Show sketches of details at a, b, and c, also plan of bracing.



PROBLEM 53.

51. Influence Diagrams for Fixed-End Moments. The method explained in Article 47 for calculating fixed-end moments can be followed for determining the ordinates to the influence diagrams for fixed-end moments of beams with either constant or variable cross section. This algebraic solution is somewhat laborious, particularly for beams with variable cross section, and for such members the following graphical solution is recommended. Before the graphical solution can be started, it is necessary that the coefficients C_1, C_2, and C_3 be calculated by the methods already described.

The basis of the graphical solution depends upon the relation between elastic curves and influence diagrams. The proof of this method for constructing influence diagrams, which was developed by Müller-Breslau, was discussed in Article 7 and will be restated briefly before its application is considered. The force system in Fig. 87a shows a load, P, which is acting at a distance x from one end, and the fixed-end moments M_{Fab} and M_{Fba}, which are to be determined. If an auxiliary force system, consisting of an applied moment M'_{ab}, Fig. 87b, which rotates the tangent at a through an angle θ_a, and an end moment, M'_{ba}, which prevents any rotation at b, acts upon the member, an elastic curve is formed which has a vertical displacement of y at any distance x from the support a. By the reciprocal theorem, which is expressed in algebraic terms by equation 5, page 10, the following relation is obtained between the forces of Fig. 87a and the displacements of Fig. 87b:

$$-M_{Fab}\theta_a + Py = 0$$

or

$$M_{Fab} = P\frac{y}{\theta_a} \tag{61}$$

The ordinates to the influence diagram for M_{Fab} of the actual force system in Fig. 87*a* are therefore equal to the ratio $\frac{y}{\theta_a}$ of the elastic curve produced by the auxiliary forces in Fig. 87*b*. A convenient graphical

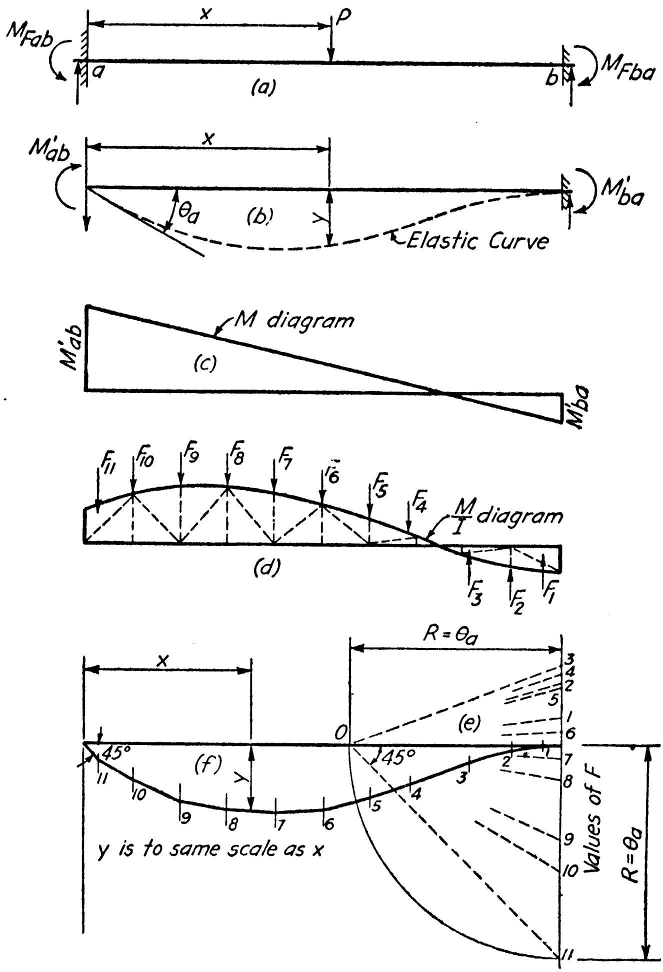


FIG. 87. Graphical solution for fixed-end moments in a beam with variable *I*.

solution for determining the ratio $\frac{y}{\theta_a}$ for the auxiliary force system will now be explained.

In terms of the beam coefficients C_1, C_2, and C_3 the end moments in Fig. 87*b* have the relation

$$M'_{ba} = \frac{C_2}{C_1} M'_{ab}$$

and the bending-moment diagram is a straight line as shown in Fig. 87*c*. The next step consists of drawing the $\frac{M}{I}$ diagram, Fig. 87*d*, as was done in the previous calculation for the coefficients, and dividing the diagram into a number of areas. According to the theorem of area moments these areas represent the change in slope of the elastic curve over a distance equal to the base of the area while the total area of the diagram is numerically equal to the end rotation θ_a. As the diagram is drawn for E equal to unity, the computed displacements are E times the actual ones; but since we are interested only in the ratio of displacements the absolute values are not necessary.

If the areas into which the $\frac{M}{I}$ diagram is divided are applied as a system of elastic weights F, a force polygon, Fig. 87*e*, can be drawn to any convenient scale, beginning with F_1 at the fixed end, b. The resultant force, R, is equal to the reaction of the conjugate beam, which in turn is equal to the end rotation θ_a. A pole distance equal to R is then measured horizontally from the point at which the force polygon is started, to give the pole O of the string polygon. If a funicular polygon, Fig. 87*f*, is drawn from the corresponding rays in the force polygon, the ordinates to this diagram are the desired ratio $\frac{y}{\theta_a}$ to the same scale as the span of the beam is drawn.

To visualize the physical meaning of the above statements it should be kept in mind that all angular rotations are small and therefore the angle in radians can be represented accurately by its tangent. For this reason any angle such as ϕ_1, Fig. 88*a*, that the tangent at point *c* makes

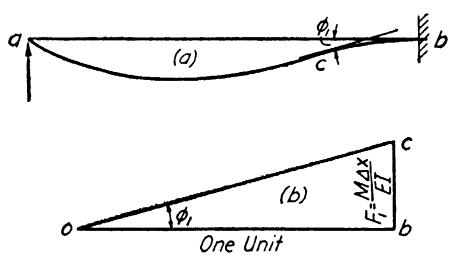


FIG. 88. Graphical determination of slope from the area of the M/EI diagram.

with the tangent at b can be obtained graphically by drawing an angle, Fig. 88*b*, whose side *bc* is equal to F, the area of the $\frac{M}{EI}$ diagram be-

tween points b and c. Now if the side bo is taken equal to θ_a instead of unity, then the angle which is still equal to the tangent is

$$\frac{F}{\theta_a} = \frac{1}{\theta_a}\phi_1 \tag{62}$$

Consequently all slopes and ordinates to the elastic curve will be multiplied by $\frac{1}{\theta_a}$ if a pole distance equal to θ_a is used in drawing the funicular polygon. The elastic curve formed by this funicular polygon is the influence diagram for M_{Fab} to the same scale as the beam is drawn. The ordinates of this elastic curve are independent of the scale to which the force polygon for the $\frac{M}{I}$ areas is drawn. As already stated, the ordinates are also independent of the value of E provided that it is constant.

Example 31. An influence diagram for the fixed-end moment M_{Fab} of the beam in Example 28, Fig. 79, will be constructed by the graphical method. The coefficients for this beam which have already been computed are

$$C_1 = 4.68 \qquad C_2 = 3.68 \qquad C_3 = 8.13$$

A moment M'_{ab} equal to 10 units is applied at a, and the moment M'_{ba} that is required to prevent rotation at b is

$$M'_{ba} = \frac{3.68}{4.68}10 = 7.86$$

The $\frac{M}{I}$ diagram for these end moments is shown in Fig. 89a, together with the areas F, into which the diagram is divided. It should be noted that the use of the triangular areas F into which the $\frac{M}{I}$ diagram is divided results in a small error in the calculations of the bending moments in the conjugate beam at the points where the F forces are applied. The bending moments in the conjugate beam and, therefore, the displacements in the actual beam are slightly high, as the moment of a small part of the $\frac{M}{I}$ diagram is neglected. The end rotation θ_a is represented by the concentrated force R which is equal to the algebraic sum of the areas, 21.37. There is no reaction at the end b of the conjugate beam, as the rotation at that end is zero. The areas are laid off to scale in the force polygon of Fig. 89b, and a pole distance equal to R or θ_a is drawn. The funicular polygon, Fig. 89c, that is drawn from the force polygon is the influence diagram for M_{Fab} to the same scale

as was used for the span L. In this problem the span L was taken equal to 10 ft, and a scale of 1 in. to 1 ft was adopted in drawing the diagram. Therefore, 1 in. of vertical distance in the influence diagram is equal to 1 ft-lb for a 10-ft span. For any span L and any load P the scale of the influence diagram will be 1 in. equals $\frac{1}{10}$ PL ft-lb.

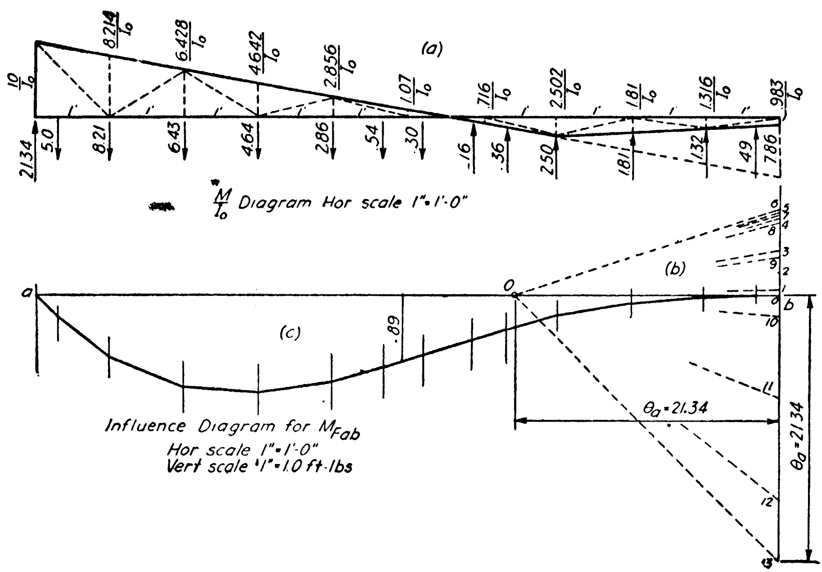


FIG. 89. Graphical solution for the influence diagram for M_{Fab} when end b is fixed.

Although the scale chosen in this problem is not large, the accuracy is equal to that obtained in the algebraic solution. The ordinate at the center of the span measured 0.89 in., and therefore the fixed-end moment for a concentrated load P at that point is $0.089PL$, which checks the value that was obtained algebraically in Example 28.

The influence diagram for M_{Fba} when the end a is simply supported is shown in Fig. 90. Here a moment of M'_{ba} equal to 10 units is applied at b and the moment M'_{ab} at a is zero. The conjugate beam which is loaded with the $\frac{M}{I}$ diagram is shown in Fig. 90a. The reactions R_1 and R_2 of the conjugate beam equal the end rotations θ_a and θ_b. Therefore, if a force polygon, Fig. 90b, is drawn for all forces acting on the conjugate beam and a pole distance equal to R_2 or θ_b is used, the resulting funicular polygon, Fig. 90c, is the influence diagram for the fixed-end moment at b when end a is free to rotate. These ordinates are again to the same scale as that used for drawing the span of the beam.

The ordinate at the center of the span is equal to $0.284PL$ as compared to the value of $0.287PL$ obtained by the algebraic solution.

The values of $\dfrac{y}{\theta_a}$, that is, the ordinates to the influence diagram, can also be obtained algebraically if desired. The displacement y is

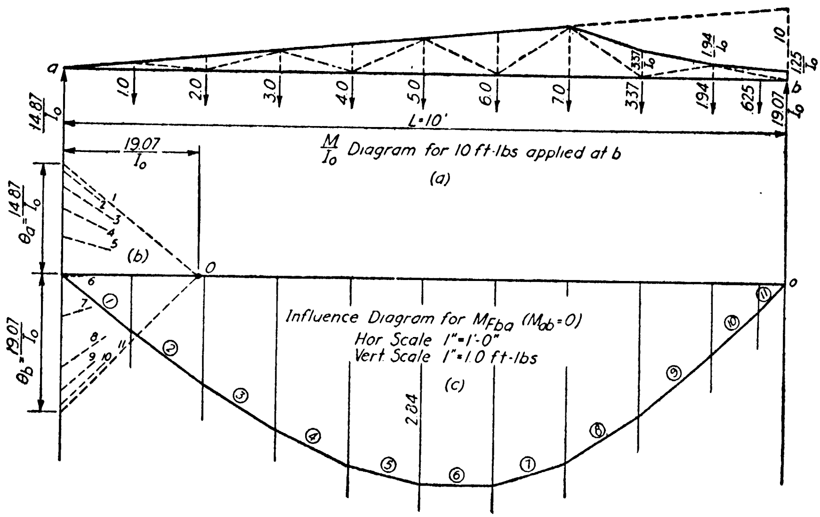


FIG. 90. Graphical solution for the influence diagram for M_{Fba} when end a is hinged.

numerically equal to the bending moment in the conjugate beam. Thus, the ordinate at the center of the span in Fig. 89 is equal to

$$\frac{y}{\theta_a} = \frac{(21.34)(5) - (1.07)(5)(2.5) - \dfrac{(8.93)(5)(2)(5)}{(2)(3)}}{21.34} = 0.886$$

The calculations for y are not difficult if a calculating machine is available.

52. Continuous Girder and Frame Bridges. Continuous girder and frame bridges can be readily analyzed by the methods of Example 30. In that example the application of the moment-distribution method to the solution of frames with members having variable I was illustrated. The only additional feature ordinarily encountered in the analysis of bridge frames that is not included in Example 30 is the calculation of stresses due to moving concentrated live loads, such as standard highway truck loading. To calculate the maximum moments and shears

for such loading it is convenient, in fact almost necessary, to construct influence diagrams for the bending moments at various sections of the frame. To fill this gap in the preceding problems the major emphasis in this discussion will therefore be placed on the numerical procedure for constructing influence diagrams for the bending moment at the ends and at intermediate sections of continuous girders and frames. In addition, some attention will be given to the calculation of internal forces due to volumetric changes that may be caused by variation in temperature or by shrinkage after the structure is completed.

The first step in the solution is the determination of the coefficients and the construction of the influence diagrams for the fixed-end moments. These calculations can be made by the algebraic or graphical methods previously explained, or, if possible, the values can be taken from curves already available, such as those prepared by the Portland Cement Association, which are given in the Appendix. It is essential, however, that a designer should be able to calculate the values, as prepared curves or tables are always limited in scope. The influence diagrams should be drawn carefully so that the ordinates at any point can be accurately scaled and also so that numerical errors may be detected.

The next step is to transform the fixed-end moments into the actual end moments, which can be accomplished almost directly by the moment-distribution method. This numerical work is greatly reduced if each fixed-end moment is treated separately and the resultant end moment then obtained from the algebraic sum of the separate values. To illustrate this procedure let us consider the three-span girder, Fig. 91, for which the distribution and carry-over factors are recorded on the diagram. The end moments are first expressed in terms of the fixed-end moment M_{Fba} by distributing a moment M_{Fba} equal to 100 as shown in Fig. 91. After the moments due to the rotation of the joints have been recorded, the end moments can be expressed in terms of the fixed-end moment M_{Fba} by the following equations:

$$M_{ba} = 0.349M_{Fba} \qquad M_{cd} = 0.182M_{Fba}$$

In the same manner, by distributing a fixed-end moment M_{Fbc} equal to 100, the end moments can be expressed in terms of M_{Fbc}. By taking advantage of the symmetry of the girder, the complete expression for the end moment M_{ba} becomes

$$M_{ba} = 0.349M_{Fba} - 0.651M_{Fbc} + 0.182(M_{Fcb} + M_{Fcd}) \quad (63a)$$

The numerical values of the fixed-end moments should be substituted into the equations for the end moments with their proper sign, that is,

positive when clockwise. By means of equation 63*a*, an influence diagram for M_{ba} can be constructed directly from the influence diagrams for the fixed-end moments.

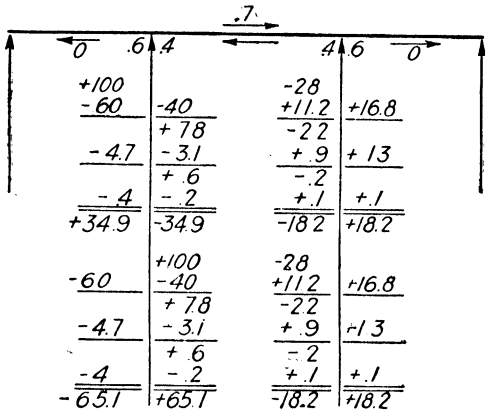


FIG. 91.

Example 32. Influence diagrams for the bending moments at several sections of a three-span continuous reinforced-concrete frame bridge, Fig. 92, will be constructed. The coefficients for the girders and abutments can be obtained from the diagrams in the Appendix; those

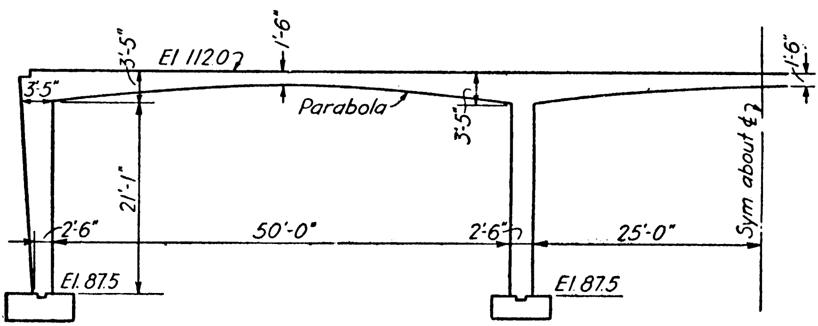


FIG. 92.

for the piers can be calculated as was done in Example 30. The coefficients, as taken from the diagrams, are

Girders

$$\frac{\min d}{\max d} = \frac{18}{47.4} = 0.415$$

for which

$$C_1 = C_3 = k = 16.1 \qquad \frac{C_2}{C_1} = C = 0.745$$

Abutments

$$\frac{\min d}{\max d} = \frac{30}{42} = 0.715$$

for which

$$C_1 = 5.2 \qquad C_3 = 8.6 \qquad C = \frac{C_2}{C_1} = 0.65$$

or

$$C_2 = (0.65)(5.2) = 3.38$$

If the abutment is assumed to be hinged at the top of the footing, the coefficient for the top of the member is (see equation 57*a*)

$$C' = C_3 - \frac{C_2^2}{C_1} = 8.6 - \frac{3.38^2}{5.2} = 6.4$$

Piers

If a value of I equal to ∞ is assumed for a distance of $0.07h$ at the top of the pier, the coefficients are

$$C_1 = 4.32 \qquad C_2 = 2.65 \qquad C_3 = 5.35$$

For the pier hinged at the top of the footing, the coefficient at the girder end is

$$C' = 5.35 - \frac{2.65^2}{4.32} = 3.72$$

The value of the distribution factors which are calculated for rectangular sections with I proportional to d^3 are recorded in Table 6.

TABLE 6

JOINT	MEMBER	C	K	CK	$r = \frac{CK}{\Sigma CK}$
b	ba	6.4	$\frac{2.5^3}{22.9}$	4.36	0.808
	bc	16.1	$\frac{1.5^3}{52.5}$	1.04	0.192
				$\Sigma CK = 5.40$	1.000
c	cb	16.1	0.0642	1.04	0.225
	cd	16.1	0.0642	1.04	0.225
	cf	3.72	$\frac{2.5^3}{22.9}$	2.53	0.550
				$\Sigma CK = 4.61$	1.000

The influence diagram, Fig. 93, for the fixed-end moment M_{Fbc} for the girder bc was drawn from the P.C.A. curves given in the Appendix. As the three girders are identical, all fixed-end moments can be

obtained from this one diagram. To obtain the end moments in terms of the several fixed-end moments the procedure already explained will

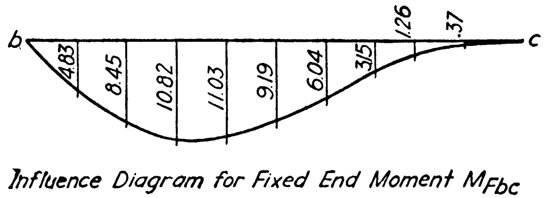


FIG. 93.

be used. In Fig. 94 the end moments have been determined for a value of 100 for each fixed-end moment by means of the moment-distribution method. Because of symmetry, the distribution need be made for only three of the six fixed-end moments. It is assumed, of course, that no translation of the joints occurs during the above calculations

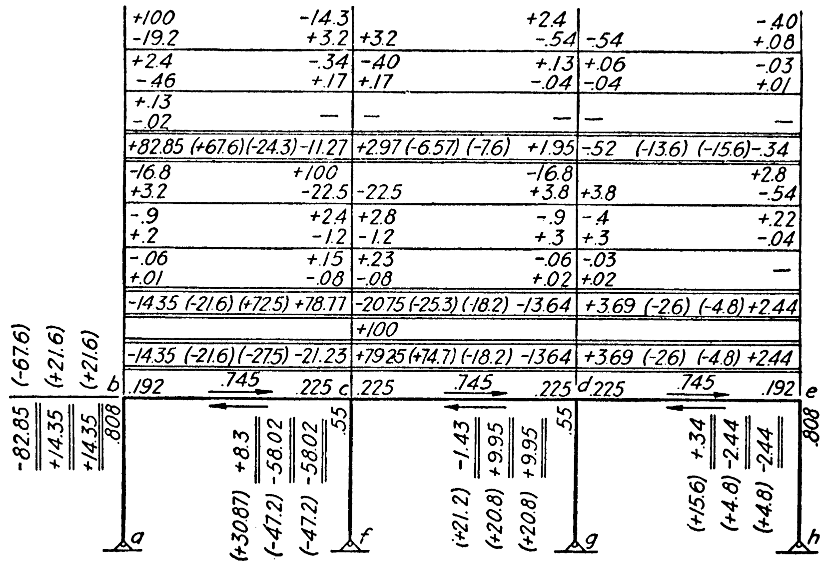


FIG. 94.

and therefore an auxiliary force F must be used to prevent any sidesway when the vertical loads are applied. For the assumption of no sidesway the moment M_{bc} can therefore be expressed by the equation

$$M_{bc} = 0.829M_{Fbc} - 0.143(M_{Fcb} + M_{Fcd}) + 0.024(M_{Fdc} + M_{Fde}) - 0.0034M_{Fed} \quad (63b)$$

and the other end moments can be expressed by similar equations.

In the above solution the auxiliary force F prevented any horizontal movement of the deck, but whether such a force will actually be developed is difficult to predict. Some horizontal resistance is undoubtedly provided by the earth pressure on the abutments and by the roadway slab, but the amount of this resistance is always uncertain. In a symmetrical structure the dead weight will cause no sidesway, and therefore for such a frame any error due to an assumption of no translation of the deck will be for the live load only. In unsymmetrical structures both dead and live loads will produce some sidesway and therefore, for such conditions, it is essential that the effect of any horizontal movement of the frame be studied.

The method for correcting for sidesway which has already been explained in Example 30 will be applied to this problem primarily to show the effect of such movement upon the influence diagrams. It will be assumed that the top of the abutments and piers move to the left a distance $E\Delta$ equal to 10 units. This movement produces fixed-end moments at the top of the vertical members equal to (see Table 6 for CK values)

$$M_{Fba} = M_{Feh} = \frac{CKE\Delta}{L} = \frac{(4.36)(10)}{22.9} = 1.9$$

$$M_{Fcf} = M_{Fdg} = \frac{(2.53)(10)}{22.9} = 1.1$$

The distribution of these fixed-end moments is recorded in Fig. 95, and, from the moments acting at the tops of the piers and abutments, the force F' is found to be

$$F' = \frac{2.38}{22.9} = 0.104$$

From the moments recorded in Fig. 94 the force F required to prevent translation of the structure when a moment M_{Fbc} equal to 100 is acting is

$$F = \frac{-82.85 + 8.3 - 1.43 + 0.34}{22.9} = -\frac{75.6}{22.9} = -3.31$$

Therefore, to remove this auxiliary force F an equal and opposite force F' must be applied which will cause end moments equal to

$$\frac{F}{F'} = \frac{3.31}{0.104} = 31.8$$

times the moments in Fig. 95. These corrections are added to the moments calculated in Fig. 94 for no sidesway, the total value being

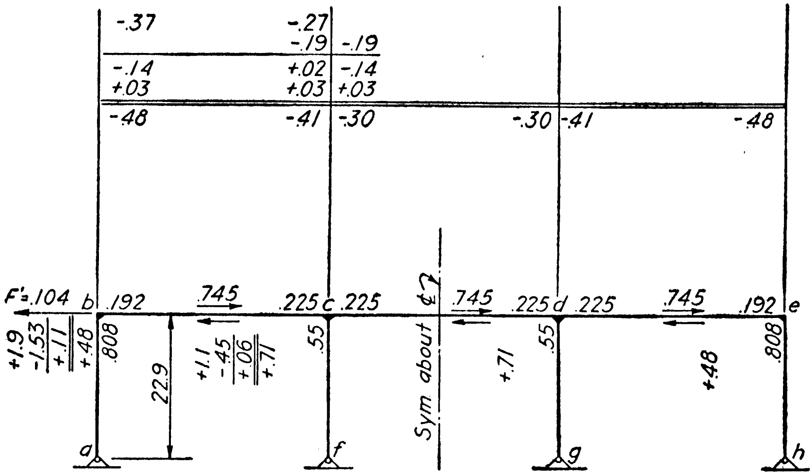


FIG. 95.

recorded in parentheses. In a similar manner the correction for sidesway has been added to all moments calculated in Fig. 94.

The equation for M_{bc} in terms of the fixed-end moments after correction for sidesway becomes

$$M_{bc} = 0.676M_{Fbc} - 0.216(M_{Fcb} + M_{Fcd}) - 0.048(M_{Fdc} + M_{Fde}) - 0.156M_{Fed}$$

The coefficients of such equations for M_{bc}, M_{cb}, M_{cd}, and M_{cf} are given in Table 7 for both sidesway (S.) and no sidesway (N.S.).

TABLE 7. COEFFICIENTS FOR END MOMENTS IN TERMS OF FIXED-END MOMENTS WITH AND WITHOUT SIDESWAY

Fixed-End Moment	M_{bc}		M_{cb}		M_{cd}		M_{cf}	
	N.S.	S.	N.S.	S.	N.S.	S.	N.S.	S.
M_{Fbc}	0.829	0.676	−0.113	−0.243	0.030	−0.065	0.083	0.308
M_{Fcb}	−0.143	−0.216	0.788	0.725	−0.207	−0.253	−0.580	−0.473
M_{Fcd}	−0.143	−0.216	−0.212	−0.275	0.793	0.747	−0.580	−0.473
M_{Fdc}	0.025	−0.048	0.037	−0.026	−0.136	−0.182	0.100	0.208
M_{Fde}	0.025	−0.048	0.037	−0.026	−0.136	−0.182	0.100	0.208
M_{Fed}	0.004	−0.156	−0.005	−0.136	0.020	−0.076	−0.014	0.212

From the coefficients in Table 7 and the fixed-end moments in Fig. 93 the influence diagrams in Fig. 96 were drawn. The effect of sidesway is clearly evident from the diagrams. There is no difference

in the areas of the two diagrams when the structure is symmetrical, and therefore the difference will be noticeable for the live load only.

After the influence diagrams have been constructed, the moment due to the dead weight of the structure can be calculated by considering the total weight as a number of concentrated loads. The algebraic sum of the product of each concentrated load and the ordinate to the

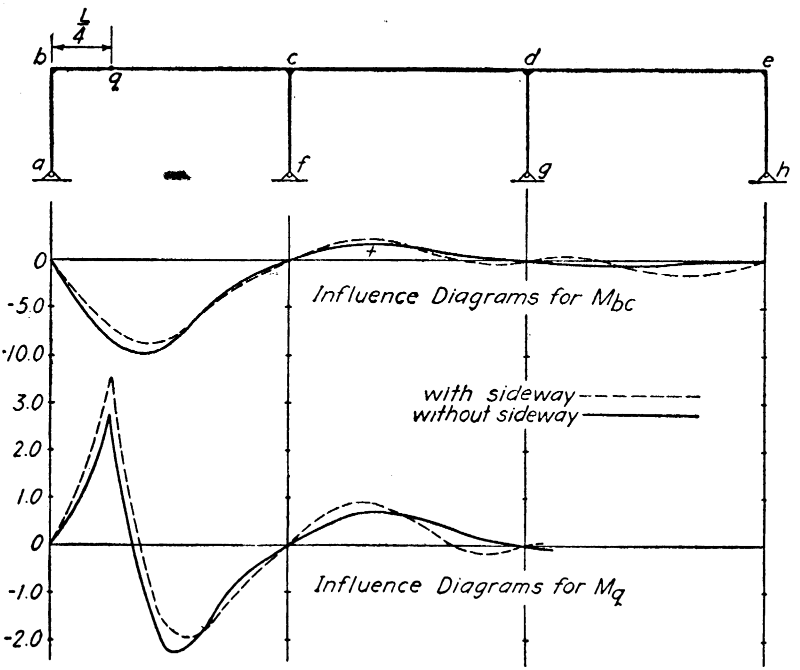


FIG. 96.

influence diagram gives the dead load moment. The same procedure is used for the concentrated live load except that the position of the loads to give a maximum value must be ascertained by trial. This operation involves no particular difficulty.

In this example the curvature of the axes of the members has been neglected. When the curvature is entirely due to increase in the depth of the members this procedure appears reasonable and gives results that are safe (see Ref. 6·4). When the curvature is more pronounced the methods described in Chapter VIII are more accurate.

53. Moments Due to Shrinkage or Temperature Change. The mathematical problem of computing the moments in continuous frames when the girders are subjected to a definite change in length will be considered in this article. It is much easier to solve this par-

ticular phase of the problem than to determine the amount of the volumetric change or the physical characteristics of the material during the movement, or to incorporate the moments into the design of the structure. These problems depend so much upon the method of construction and the effect of local overstressing upon both main members and connections that no single procedure is desirable for all structures. In general, the amount of the change in length and the effect upon the structure should be anticipated as accurately as possible. For reinforced concrete a nominal coefficient of shrinkage of $0.0002L$ is frequently employed, although the actual contraction will depend upon many factors, such as consistency of the concrete when poured, method of construction, humidity, and temperature. Coefficients of linear expansion of 6×10^{-6} for concrete and 6.5×10^{-6} for steel structures are common values. The stresses produced by shrinkage and temperature changes depend upon the structural arrangement and relative stiffness of the members. The calculation of such stresses follows the same procedure as was employed in the analysis of continuous frames subject to translation or sidesway.

After the change in length of each girder has been estimated, the relative movement of the top of each abutment or pier can be determined with respect to some point that is assumed to remain stationary. If the structure is symmetrical the axis of symmetry, of course, provides an actual fixed point and any displacement with respect to it represents an actual movement. The problem then consists simply of distributing the fixed-end moments in the piers and abutments that are computed from the actual displacements with respect to the center line.

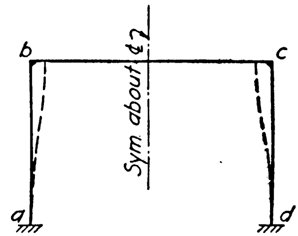


FIG. 97.

When the structure is unsymmetrical, some point, preferably the top of one of the piers or an abutment, must be assumed to be stationary, and the relative motion of the other joints is determined with respect to it. This configuration of the structure requires an auxiliary horizontal force, F, applied at the assumed reference point, to maintain equilibrium of the frame. This auxiliary force is then removed in the manner already explained. Thus, for the symmetrical single-span frame in Fig. 97, one-half of the total change in length Δ of the deck should be assigned to each joint with the motion toward the center. If the structure is unsymmetrical, Fig. 98a, one joint, such as b, should be assumed fixed and the entire movement of the deck should be assigned to joint c. For this condition there are fixed-end moments in

member cd only, and an auxiliary force, F, is applied at b. This auxiliary force is then removed by applying an equal and opposite force as in Fig. 98b. The same conditions are shown for a three-span frame

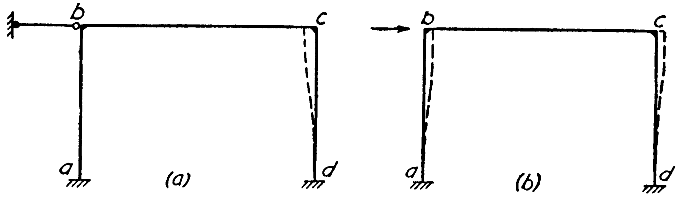


FIG. 98.

in Fig. 99. The displacements for a symmetrical frame are illustrated in Fig. 99a, and the relative displacements to be used for an unsymmetrical frame are given in Figs. 99b and c.

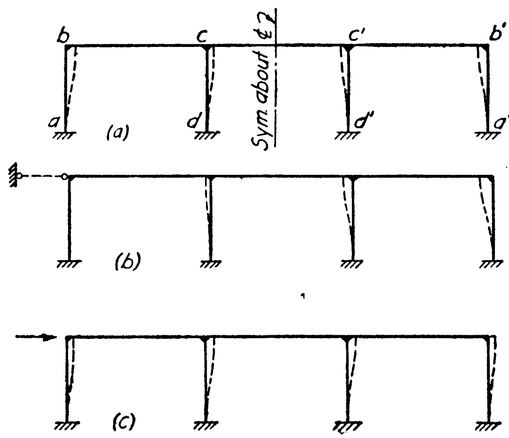


FIG. 99.

Example 33. The moments in the three-span reinforced-concrete frame of Fig. 92 will be calculated for a shrinkage of $0.0002L$ and a temperature drop of 45° F. A coefficient of linear expansion of 6×10^{-6} and a value of E of 2.5×10^6 are used.

The total change per unit of length of the deck is

$$0.0002 + (45)(0.000006) = 0.00047$$

As the structure is symmetrical, the tops of the piers and abutments move towards the center with the following displacements:

Top of piers

$$(0.00047)(26.25) = 0.0123 \text{ ft}$$

Top of abutments

$$(0.00047)(78.75) = 0.037 \text{ ft}$$

The fixed-end moments caused by these displacements are

$$M_{Fba} = -M_{Feh} = -\frac{(4.36)(2500)(144)(0.037)}{(12)(22.9)} = -212 \text{ ft-kips}$$

$$M_{Fcf} = -M_{Fdg} = -\frac{(2.53)(2500)(144)(0.0123)}{(12)(22.9)} = -41.1 \text{ ft-kips}$$

The coefficient $\frac{1}{12}$ was restored to the moment of inertia as it was omitted from the CK values in Table 6.

The distribution of the above fixed-end moments, which follows the usual procedure, is tabulated in Fig. 100. Only half of the structure is

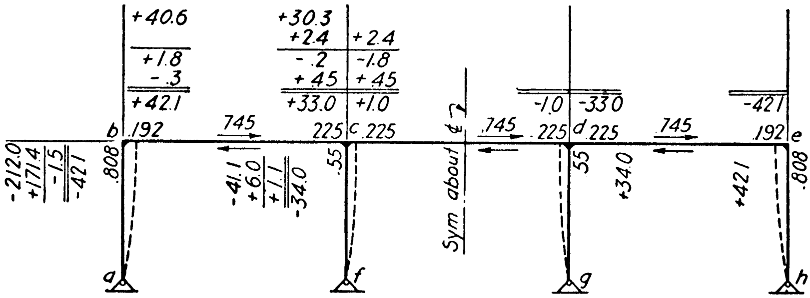


FIG. 100.

shown, as the corresponding moments on the right half are equal but of opposite sign. It is important to note that both end moments of the end girders are clockwise and consequently reduce the dead-load moment at the abutment end but increases the value at the pier end. This distribution differs from that for a single span in which the dead-load moments are reduced at both ends. In contrast to this action the positive dead-load bending moment at the center of the girders is increased but slightly in the three-span frame, whereas, for the single span, the center moment increases by the amount of the end moments.

Example 34. The three-span frame analyzed in Example 33 by utilizing the symmetry of the structure will now be analyzed by the procedure required for unsymmetrical structures.

The top of the first pier, point c, will be chosen as a fixed point, so that the relative displacements of the other points with respect to it are

Point b	$(0.00047)(52.5)$	$= 0.0247$
Point d	$(0.00047)(52.5)$	$= 0.0247$
Point e	$(0.00047)(105.0)$	$= 0.0494$

The fixed-end moments due to these displacements are

$$M_{Fba} = -141.7 \text{ ft-kips} \qquad M_{Fcf} = 0$$

$$M_{Fdg} = 82.2 \text{ ft-kips} \qquad M_{Feh} = 283.4 \text{ ft-kips}$$

The distribution of these fixed-end moments is recorded in Fig. 101, and the auxiliary force F necessary to maintain this configuration of the frame is

$$F = \frac{87.1}{22.9} = 3.81 \text{ kips}$$

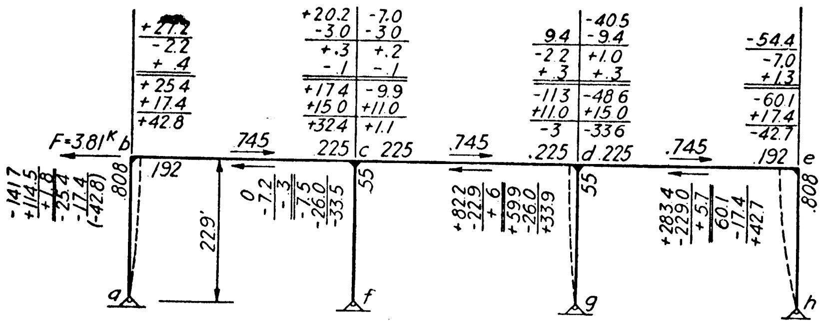


FIG. 101.

To remove this force, an opposite force such as F' of Fig. 95 must be applied, and, as the numerical value of F' must be equal to F, the moments in Fig. 95 must be multiplied by

$$\frac{F}{F'} = \frac{3.81}{0.104} = 36.6$$

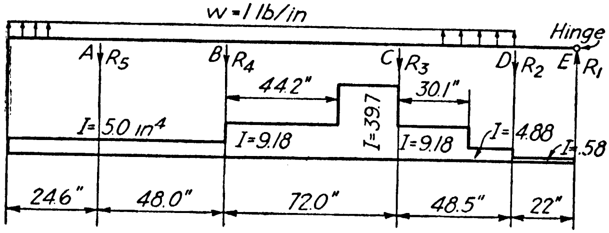
and added to the moments in Fig. 101. These values, as well as the final results, are shown in Fig. 101. The discrepancies between the final values in Figs. 100 and 101 are due to differences in the rate of convergence and order of distribution of the two methods as two cycles of distribution were used in each problem.

PROBLEMS

54. (*a*) Construct an influence diagram for M_{Fab} for member *ab* in Problem 51 by the graphical method when end *b* is assumed fixed.

(*b*) Construct an influence diagram for M_{Fba} for member *ab* when end *a* is assumed hinged.

55. Determine the moments at the supports of the continuous elevator spar shown in the figure. Calculate the coefficients and fixed-end moments for the values of I as shown. Assume that the supports have no vertical movement.

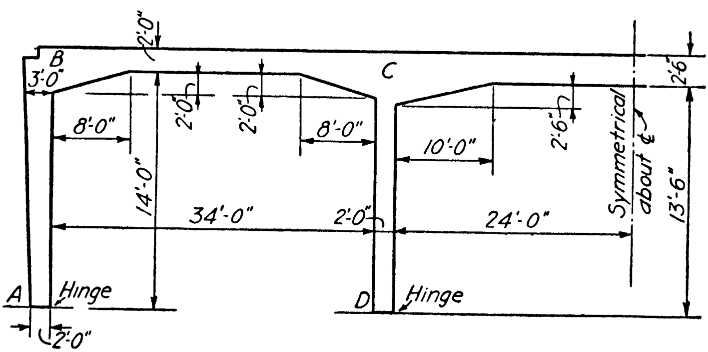


PROBLEM 55.

56. Calculate the moments in the elevator spar of Problem 55 due to the following relative displacements of the horizontal stabilizer to which the elevator is attached:

$\Delta_{AB} = 2.78$ in. $\quad \Delta_{BC} = 3.46$ in. $\quad \Delta_{CD} = 1.50$ in. $\quad \Delta_{DE} = 0.4$ in.

57. Construct an influence diagram for the moment at B and at the center of span BC for the continuous frame shown. Assume a width of 12 in. for all members.



PROBLEM 57.

58. Calculate the end moments for all members of the continuous frame in Problem 57 for a temperature drop of 60°. Use a coefficient of linear expansion of 6×10^{-6} and a modulus of elasticity of 2.5×10^6 lb per in.^{2}

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CHAPTER VII

CONTINUOUS TRUSSES AND BENTS

54. Truss Deflections. Both algebraic and graphical methods that are frequently used for calculating truss deflections were explained in Chapter II. The algebraic methods include: (*a*) equality of external and internal work (Article 10); (*b*) Castigliano's theorem (Article 11). As the equations and numerical operations for these two methods are identical in their final form, either can be used in the subsequent analysis. Important advantages in the application of Castigliano's theorem are the directness and simplicity with which the necessary equations are derived and expressed. When the internal-strain energy is expressed in terms of both known and unknown force systems a convenient arrangement is provided for the calculation of either displacements or redundant forces.

The graphical solution most frequently employed is the application of the Williot diagram for determining relative displacements (Article 16), together with the Mohr rotation diagram to give absolute displacements (Article 17). These two diagrams provide an exceedingly flexible and accurate tool for the calculation of truss deflections. Moreover, they have the advantage of giving all displacements in one solution whereas the algebraic methods require the calculation of each displacement separately. If the final displacement obtained by the graphical solution is checked algebraically, all values in the displacement diagrams must be correct. Combining the Williot and Mohr diagrams with the reciprocal theorem (Article 7) provides a convenient method for the construction of influence diagrams for redundant forces; it is similar to the method applied previously to the calculation of fixed-end moments in beams.

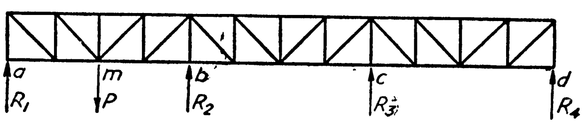


FIG. 102.

55. Algebraic Procedure. The equations for calculating the deflections in any truss, such as in Fig. 102, are readily obtained by

Castigliano's theorem. The force system is replaced for analytical purposes by the three separate force systems in Figs. 103*a*, *b* and *c*. If the stresses due to these force systems are combined, the total stress in any member is equal to

$$S = S' + R_1u_1 + R_2u_2 \tag{64}$$

in which S' is the stress due to load P, Fig. 103*a*,
u_1 is the stress due to R_1 equal to unity, Fig. 103*b*,
u_2 is the stress due to R_2 equal to unity, Fig. 103*c*.

The total strain energy in the truss is therefore equal to

$$U = \sum \frac{S^2L}{2AE} = \sum \frac{(S' + R_1u_1 + R_2u_2)^2L}{2AE} \tag{65}$$

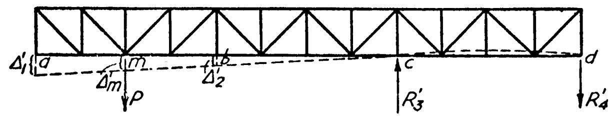


FIG. 103*a*.

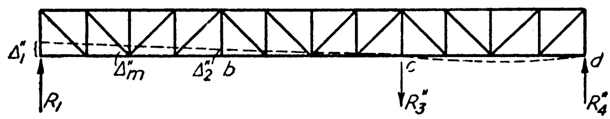


FIG. 103*b*.

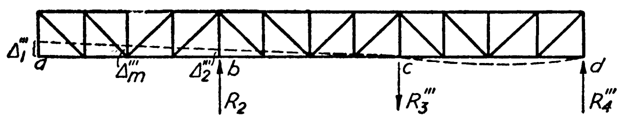


FIG. 103*c*.

By means of equations 65 and 13 (Article 11), the displacements at any point in a truss can be calculated. Thus, the displacement Δ_1 in the direction of R_1 is equal to

$$\Delta_1 = \frac{\partial U}{\partial R_1} = \sum \frac{(S' + R_1u_1 + R_2u_2)u_1L}{AE} = \sum \frac{Su_1L}{AE} \tag{66a}$$

and, in the same manner,

$$\Delta_2 = \frac{\partial U}{\partial R_2} = \sum \frac{(S' + R_1u_1 + R_2u_2)u_2L}{AE} = \sum \frac{Su_2L}{AE} \tag{66b}$$

$$\Delta_m = \frac{\partial U}{\partial P} = \sum \frac{(S' + R_1u_1 + R_2u_2)u'L}{AE} = \sum \frac{Su'L}{AE} \tag{66c}$$

In equation 66*c*, u' is the stress in the member of the simply supported truss for P equal to unity, and with R_1 and R_2 regarded as part of the applied forces. Equation 66*c* can also be written so as to regard R_1 and R_2 as functions of P, that is

$$\Delta_m = \frac{\partial U}{\partial P} = \sum \frac{S \dfrac{\partial S}{\partial P} L}{AE} = \sum \frac{SuL}{AE} \tag{66d}$$

where u is the resultant stress S in the member of the continuous truss for P equal to unity. Although u' is not equal to u, the summations of the terms in equations 66*c* and 66*d* are the same as it is immaterial whether the truss is regarded as a simply supported truss subjected to applied loads of P, R_1, and R_2 or as a continuous truss subjected to the load P.

If the displacement is desired at any point at which a load is not applied, an auxiliary or virtual force P_x, applied at the point, must be included in the forces acting on the structures.

For such a combined force system, the total stress S in any member is

$$S = S' + R_1u_1 + R_2u_2 + S_x \tag{67}$$

in which S_x is the stress due to P_x, the auxiliary force.

The displacement Δ at the point where P_x is applied is

$$\Delta_x = \frac{\partial U}{\partial P_x} = \sum \frac{(S' + R_1u_1 + R_2u_2 + S_x)u_xL}{AE}$$

but, since the value of S_x is actually zero,

$$\Delta_x = \sum \frac{Su_xL}{AE} \tag{68}$$

in which S is the stress due to the actual load and u_x is the stress due to an auxiliary load of unity applied at point x on the *simply supported* structure.

All the above displacements can be obtained by combining the separate displacements for each force system in Fig. 103. For example, the displacement Δ_m is equal to the sum

$$\Delta_m = \Delta_m' + \Delta_m'' + \Delta_m''' = \sum \frac{S'u'L}{AE} + \sum \frac{R_1u_1u'L}{AE} + \sum \frac{R_2u_2u'L}{AE}$$

which is identical with equation 66*c*.

56. Redundant Reactions. The reactions of continuous trusses are frequently calculated from strain equations obtained by assuming that

the supports undergo no vertical displacements. In Fig. 102, if the displacements at R_1 and R_2 are equal to zero, equations 66*a* and 66*b* can be written in the form

$$\Delta_1 = \frac{1}{E}\left\{\sum \frac{S'u_1L}{A} + R_1\sum \frac{u_1^2L}{A} + R_2\sum \frac{u_1u_2L}{A}\right\} = 0 \quad (69a)$$

$$\Delta_2 = \frac{1}{E}\left\{\sum \frac{S'u_2L}{A} + R_1\sum \frac{u_1u_2L}{A} + R_2\sum \frac{u_2^2L}{A}\right\} = 0 \quad (69b)$$

If the constant terms in the above equations are represented by

$$C_1 = \sum \frac{u_1^2L}{A} \qquad C_{12} = \sum \frac{u_1u_2L}{A} \qquad C_2 = \sum \frac{u_2^2L}{A}$$

$$C' = \sum \frac{S'u_1L}{A} \qquad C'' = \sum \frac{S'u_2L}{A}$$

the reactions R_1 and R_2 are given by the following expressions:

$$R_1 = \frac{C''C_{12} - C'C_2}{C_1C_2 - C_{12}^2} \qquad (70a)$$

$$R_2 = \frac{C'C_{12} - C''C_1}{C_1C_2 - C_{12}^2} \qquad (70b)$$

Similar equations can be established for a continuous truss of any number of spans. For a four-span continuous truss, another redundant reaction, say R_3, must be used, and the following additional constant terms computed:

$$C_3 = \sum \frac{u_3^2L}{A} \qquad C_{13} = \sum \frac{u_1u_3L}{A} \qquad C_{23} = \sum \frac{u_2u_3L}{A}$$

$$C''' = \sum \frac{S'u_3L}{A}$$

For a continuous truss of four spans that has no vertical displacements at the supports the following equations can be written

$$\Delta_1 = C_1R_1 + C_{12}R_2 + C_{13}R_3 + C' = 0 \qquad (71a)$$

$$\Delta_2 = C_{12}R_1 + C_2R_2 + C_{23}R_3 + C'' = 0 \qquad (71b)$$

$$\Delta_3 = C_{13}R_1 + C_{23}R_2 + C_3R_3 + C''' = 0 \qquad (71c)$$

from which the values of R_1, R_2, and R_3 can be computed.

57. Influence Diagrams for Reactions. A semi-graphical method for obtaining the influence diagram for any redundant reaction of a continuous truss is often superior to the algebraic procedure just explained. The reciprocal theorem forms the basis of the method, while the Williot and Mohr displacement diagrams provide a convenient graphical solution for calculating the necessary deflections. If some force R_1' is applied to a truss as in Fig. 104, and the elastic curve of the structure drawn, then it will be found from the reciprocal theorem that the reaction R_1, Fig. 102, for any applied load P is equal to

$$R_1 = P\frac{y}{\Delta_1} \tag{72}$$

Before the values of y and Δ_1 can be obtained, the reactions and stresses due to R_1', Fig. 104, must be known. This part of the solution

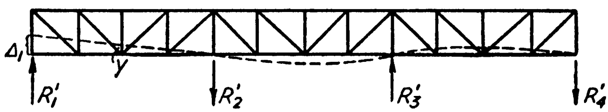


FIG. 104.

is best made by the algebraic methods, as the calculations are not difficult and also they provide a check upon the graphical solution. From equation 66*b*, with S' equal to zero, the relation between R_1' and R_2' to make Δ_2 equal to zero is

$$R_2' = -R_1'\frac{\sum \frac{u_1u_2L}{A}}{\sum \frac{u_2^2L}{A}} \tag{73}$$

After equation 73 is solved, the stresses S in the members and the change in length of each member, $\frac{SL}{AE}$, are computed and the displacement diagrams drawn. The length L can be taken in foot units and $\frac{1}{E}$ omitted for convenience as only the ratios of displacements are used in the solution. A check upon all displacements can be obtained by computing Δ_1 from the equation

$$\Delta_1 = \frac{1}{E}\sum u_1\frac{SL}{A} \tag{74}$$

This check on the results of the graphical solution can be quickly made as the terms in equation 74 are already known.

Example 35. The ordinates to the influence diagram for the reaction R_1 of the three-span continuous truss in Fig. 105 will be calculated by the semi-graphical method. The lengths and areas of the members

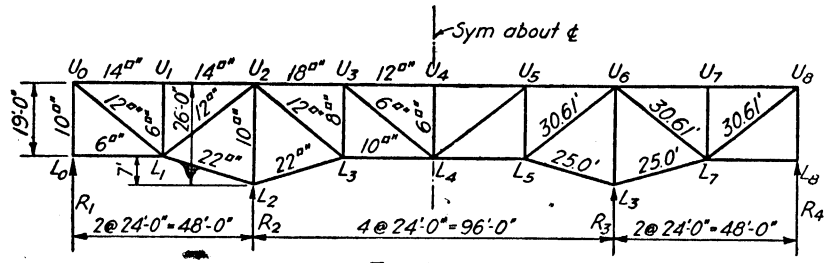


FIG. 105.

are recorded on the truss diagram. Values of u_1 and $\dfrac{u_1L}{A}$ for R_1 equal to unity and of u_2 and $\dfrac{u_2L}{A}$ for R_2 equal to unity are given in Figs. 106*a* and 106*b*, respectively. By means of equation 73, the reaction R_2'

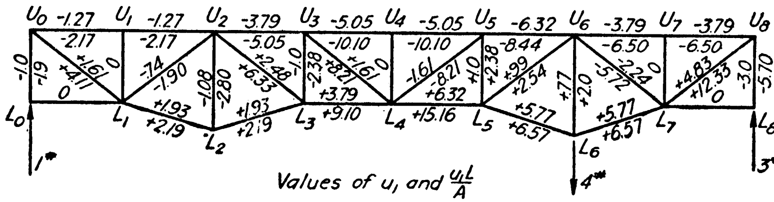


FIG. 106*a*.

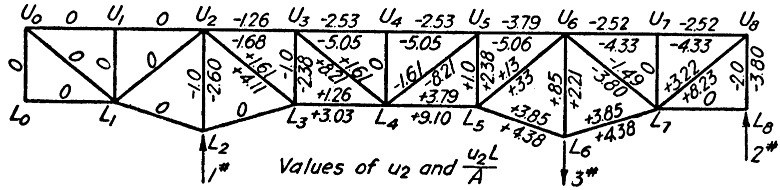


FIG. 106*b*.

is obtained directly from the u_1, u_2, and $\dfrac{u_2L}{A}$ quantities for any applied force R_1'. The summations of these terms for all members of the truss are

$$\sum \frac{u_1u_2L}{A} = 347.5 \qquad \sum \frac{u_2^2L}{A} = 222.7$$

Substituting these quantities in equation 73 gives the value of R_2' as

$$R_2' = -\left(\frac{347.5}{222.7}\right) R_1' = -1.56R_1'$$

For an applied load R_1' equal to unity the reactions that will prevent any vertical displacement at L_2, L_6, and L_8 and also satisfy the equilibrium requirements are

$$R_2' = -1.56 \qquad R_3' = 0.677 \qquad R_4' = -0.118$$

The stresses S in the truss members for R_1' equal to unity are therefore equal to

$$S = u_1 - 1.56u_2$$

and the change in length of each member (times E) is

$$\frac{SL}{A} = \frac{u_1L}{A} - 1.56\frac{u_2L}{A}$$

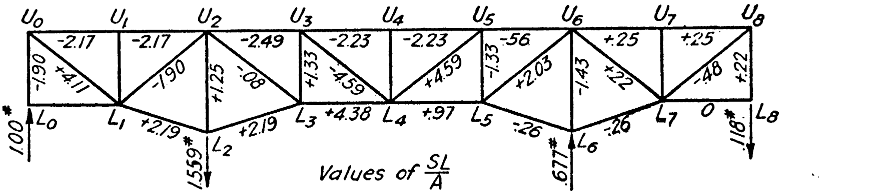
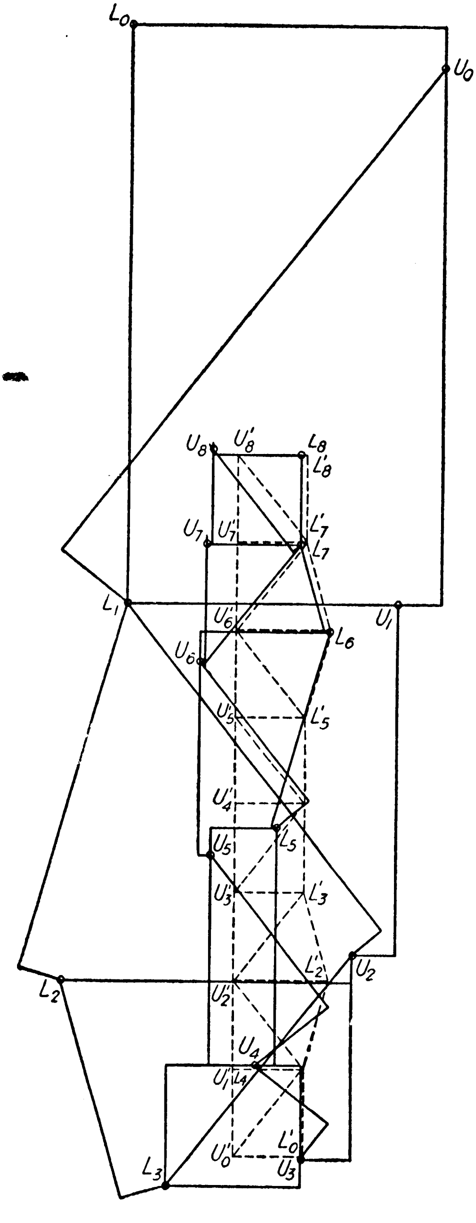


FIG. 106c.

The numerical values of $\frac{SL}{A}$ as recorded on the truss diagram in Fig. 106c were used to draw the Williot diagram in Fig. 107. In this diagram L_4 was selected as the fixed point and member U_4L_4 as a fixed axis. The construction of the displacement diagram which follows the procedure explained in Article 16 should be executed accurately on a large drawing board with good equipment. The position of each point should be back-checked as soon as it is determined to avoid extending any error to succeeding points.

After the Williot diagram is completed it is apparent that U_4L_4 is not a fixed axis, as such an assumption gives a relative vertical displacement of points L_2, L_6, and L_8. Consequently the structure is rotated about L_6 as a fixed point until the displacement at L_2 is zero. The displacements due to the rotation are given by the Mohr rotation diagram, which is indicated by broken lines. As the displacement of L_8 must also be zero a check on the accuracy of the drawing work is obtained. The vertical displacement of any point is equal to the



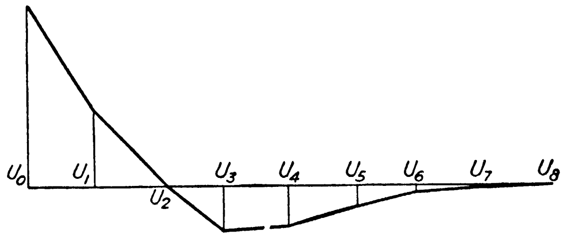
Displacement Diagram
For A Continuous Truss
Scale 1" = 4 units

FIG. 107.

vertical distance from the point on the Mohr diagram to the corresponding point on the Williot diagram. The vertical movement of L_0 is equal to the vertical component of the vector $L_0'L_0$ or 55.0 units, and the vertical displacement of U_1 is 22.5 units. The value of R_1 in Fig. 105 for a unit load at U_1 is, therefore,

$$R_1 = \frac{y}{\Delta_a} = \frac{22.5}{55.0} = 0.409$$

The vertical displacement of all upper panel points and the corresponding ordinates to the influence diagram are tabulated in Table 8. The influence diagram in Fig. 108 is drawn from these values.



Influence Diagram for R_1

FIG. 108.

TABLE 8

POINT	VERTICAL DISPLACEMENT	VALUE OF R_1 $\frac{y}{55.0}$
U_0	+53.1	+0.965
U_1	+22.5	+0.409
U_2	+1.2	+0.022
U_3	−12.9	−0.235
U_4	−12.6	−0.229
U_5	−6.6	−0.120
U_6	−1.4	−0.025
U_7	−0.1	−0.002
U_8	+0.2	+0.004

A check on the above values is obtained by computing the displacement of L_0 algebraically from equation 74, which gives

$$\Delta_1 = \frac{1}{E}\sum u_1 \frac{SL}{A} = \frac{55.0}{E}$$

which is identical with the value obtained from the Williot diagram.

58. End Forces and Couples. The forces acting upon any span of a continuous truss, such as span ab, Fig. 109, are analogous to those in a continuous beam inasmuch as they consist of the applied loading, end shears, and end couples. Moreover, as in the analysis of continuous beams, any span ab, Fig. 109, can first be assumed as fixed at the ends

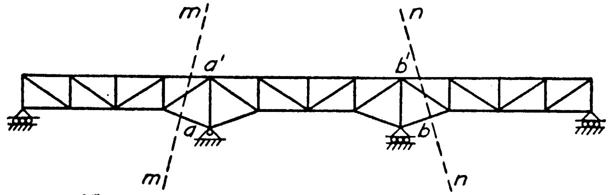


FIG. 109.

and these fixed-end forces afterwards corrected so as to provide the necessary equilibrium and strain conditions between continuous spans. The horizontal components or end couples at a and a', also b and b', must balance, and the horizontal displacement of these points must be the same for both adjacent spans.

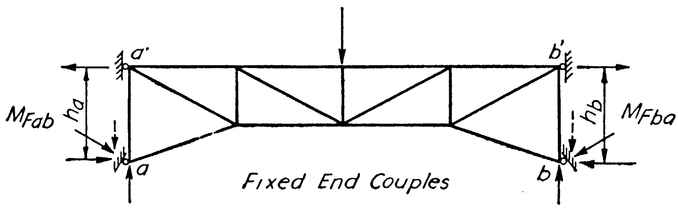


FIG. 110a.

To illustrate the above statements graphically, the fixed-end forces in Fig. 110a must be corrected by adding the end forces in Figs. 110b and c so as to make the end rotations

$$\theta_a = \frac{\Delta_a + \Delta_{a'}}{h_a} \quad \text{and} \quad \theta_b = \frac{\Delta_b + \Delta_{b'}}{h_b}$$

equal for both spans. These end forces or couples can be expressed in terms of the rotation θ_a and θ_b, as was done for the end couples applied

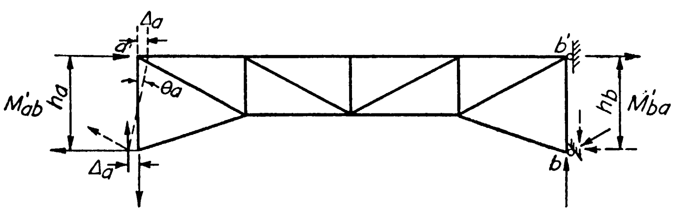


FIG. 110b.

to continuous beams. All three problems represented by Figs. 110*a*, *b*, and *c* are statically indeterminate problems, each of which can be solved by the methods already explained. The solution of these problems gives the fixed-end couples, stiffness factors, and carry-over factors—quantities that correspond to the same terms used in the moment-distribution method for continuous beams.

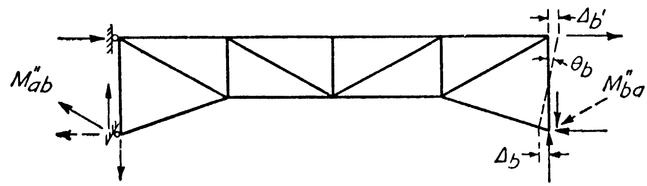


FIG. 110*c*.

59. Distribution and Carry-Over Factors. The determination of the end forces is best illustrated by calculating the end couple M'_{ba}, Fig. 110*b*, that is required to hold joints b and b' without horizontal translation when some end couple, M'_{ab}, is applied at end a. The total strain energy U in the truss for this force system is equal to

$$U = \sum \frac{(M'_{ab}u_a + M'_{ba}u_b)^2 L}{2AE} \tag{75}$$

in which u_a equals stress for $M'_{ab} = 1, \quad M'_{ba} = 0.$

u_b equals stress for $M'_{ab} = 0, \quad M'_{ba} = 1.$

The end rotations θ_a and θ_b, as given by Castigliano's theorem, are

$$\theta_a = \frac{\partial U}{\partial M'_{ab}} = \sum \frac{(M'_{ab}u_a + M'_{ba}u_b)u_a L}{AE} \tag{76a}$$

$$\theta_b = \frac{\partial U}{\partial M'_{ba}} = \sum \frac{(M'_{ab}u_a + M'_{ba}u_b)u_b L}{AE} \tag{76b}$$

When θ_b is equal to zero, equation 76*b* gives

$$M'_{ba} = -\frac{\sum \frac{u_a u_b L}{A}}{\sum \frac{u_b^2 L}{A}} M'_{ab} = C_{ab} M'_{ab} \tag{77}$$

where

$$C_{ab} = -\frac{\sum \frac{u_a u_b L}{A}}{\sum \frac{u_b^2 L}{A}} \tag{78}$$

is the carry-over factor from a to b. As the term $\sum \frac{u_a u_b L}{A}$ is negative the value of C_{ab} is positive.

If the value of M'_{ba} from equation 77 is now substituted in equation 76a the expression for θ_a becomes

$$\theta_a = \sum \frac{(M'_{ab}u_a + C_{ab}M'_{ab}u_b)u_a L}{AE} \tag{79a}$$

from which

$$M'_{ab} = \frac{E\theta_a}{\sum \frac{u_a^2 L}{A} + C_{ab}\sum \frac{u_a u_b L}{A}} = C_a E\theta_a \tag{79b}$$

where

$$C_a = \frac{1}{\sum \frac{u_a^2 L}{A} + C_{ab}\sum \frac{u_a u_b L}{A}} \tag{79c}$$

is the stiffness factor and corresponds to C_1K_0 or C_3K_0 for a beam.

The end couple M''_{ab}, Fig. 110c, that will hold joints a and a' fixed when a couple M''_{ba} is applied at end b, is calculated in a similar manner. The necessary equations are obtained by interchanging the subscripts a and b in equations 77, 78, and 79, or

$$M''_{ab} = C_{ba}M''_{ba} \tag{80a}$$

in which

$$C_{ba} = -\frac{\sum \frac{u_a u_b L}{A}}{\sum \frac{u_a^2 L}{A}} \tag{80b}$$

is the carry-over factor from b to a. Also,

$$M''_{ba} = C_b E\theta_b \tag{81a}$$

where

$$C_b = \frac{1}{\sum \frac{u_b^2 L}{A} + C_{ba}\sum \frac{u_b u_a L}{A}} \tag{81b}$$

After the coefficients C_a and C_b have been computed for each span the distribution factors r can be determined, as

$$r_{ab} = \frac{C_a}{\Sigma C_a} \quad \text{and} \quad r_{ba} = \frac{C_b}{\Sigma C_b} \tag{82}$$

or the ratio of the coefficient of one span to the sum of the coefficients for both spans.

Equation 79*c* can be used when the truss is simply supported at *b* by making u_b equal to zero, or

$$C_a = \frac{1}{\sum \frac{u_a^2 L}{A}} \tag{83a}$$

When the truss is simply supported at *a*, the coefficient C_b becomes

$$C_b = \frac{1}{\sum \frac{u_b^2 L}{A}} \tag{83b}$$

60. Fixed-End Couples. Either the algebraic or preferably the semi-graphic method can be used to calculate the end couples required to hold points a, a', b, b' without translation when any load is applied on the span. The stress in any member is equal to

$$S = S' + M_{Fab}u_a + M_{Fba}u_b \tag{84}$$

where S' = stress due to applied load on a simply supported span.
u_a, u_b = stress due to unit end couples at a and b, respectively.

The total strain energy in the truss is

$$U = \sum \frac{S^2 L}{2AE} = \sum \frac{(S' + M_{Fab}u_a + M_{Fba}u_b)^2 L}{2AE} \tag{85}$$

and the end rotations θ_a and θ_b equal

$$\theta_a = \frac{\partial U}{\partial M_{Fab}}$$

$$= \frac{1}{E}\left\{\sum \frac{S'u_a L}{A} + M_{Fab}\sum \frac{u_a^2 L}{A} + M_{Fba}\sum \frac{u_a u_b L}{A}\right\} = 0 \tag{86a}$$

$$\theta_b = \frac{\partial U}{\partial M_{Fba}}$$

$$= \frac{1}{E}\left\{\sum \frac{S'u_b L}{A} + M_{Fab}\sum \frac{u_a u_b L}{A} + M_{Fba}\sum \frac{u_b^2 L}{A}\right\} = 0 \tag{86b}$$

The fixed-end couples can be obtained by solving equations 86*a* and 86*b* simultaneously. These equations are similar to equations 69*a* and *b* except that the summation is for one span only instead of the entire structure.

Influence diagrams for the fixed-end moments or couples in a continuous truss are best obtained directly from an elastic curve of the structure by applying the Müller-Breslau principle as in Article 51 for the fixed-end moments in continuous beams. The procedure is similar to that followed in other problems that have been solved by means of the reciprocal theorem. If some end couple M'_{ab} is applied at end a, Fig. 110b, the couple M'_{ba} required to hold end b is, from equation 77,

$$M'_{ba} = C_{ab}M'_{ab}$$

The stresses S and $\frac{SL}{A}$, the change in length times E, must be determined for any assumed value of M'_{ab}. If M'_{ab} is taken equal to unity

$$S = u_a + C_{ab}u_b$$

$$\frac{SL}{A} = \frac{u_aL}{A} + C_{ab}\frac{u_bL}{A}$$

The displacements y and rotation θ_a are given directly by a Williot diagram drawn for the $\frac{SL}{A}$ values if point b is selected as a fixed point and bb' as a fixed axis. The ordinates to the influence diagram for M_{Fab}, as in other problems, are equal to

$$M_{Fab} = P\frac{y}{\theta_a} = Ph_a\left(\frac{y}{\Delta_a + \Delta'_a}\right) \tag{87}$$

Example 36. The end moments and reactions of the continuous truss of Example 35, Fig. 105, will be determined for a unit load at point U_4 by the moment-distribution method.

The stiffness coefficient C_b for the end spans is given by equation 83b as the spans are simply supported at L_0 and L_8. The values of u_b and $\frac{u_bL}{A}$ for a unit end moment applied to the end span are recorded in Fig. 111a. From these values, the coefficient C_{b1} is equal to

$$C_{b1} = \frac{1}{\sum \frac{u_b^2L}{A}} = \frac{1}{0.00877} = 114$$

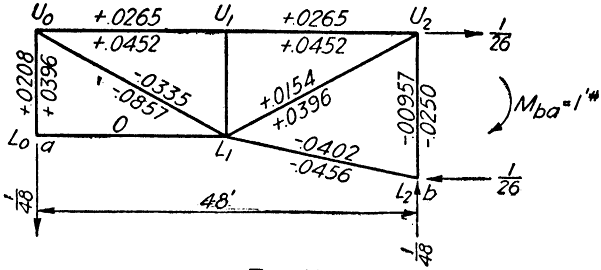


FIG. 111*a*.

The values of u_b, $\dfrac{u_bL}{A}$, u_c, and $\dfrac{u_cL}{A}$ are recorded in Figs. 111*b* and *c* for positive unit end couples applied to the center span. From these quantities, the coefficients C_{b2}, C_{bc}, and C_{c2} are found to be

$$C_{bc} = C_{cb} = -\frac{\sum \dfrac{u_bu_cL}{A}}{\sum \dfrac{u_c^2L}{A}} = -\frac{-0.00343}{0.0155} = 0.221$$

$$C_{b2} = C_{c2} = \frac{1}{\sum \dfrac{u_b^2L}{A} + C_{bc}\sum \dfrac{u_bu_cL}{A}}$$

$$= \frac{1}{0.0155 + (0.221)(-0.00343)} = 68$$

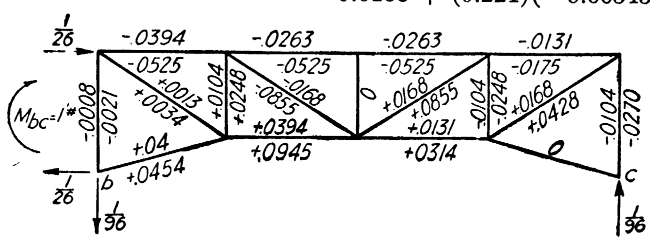


FIG. 111*b*.

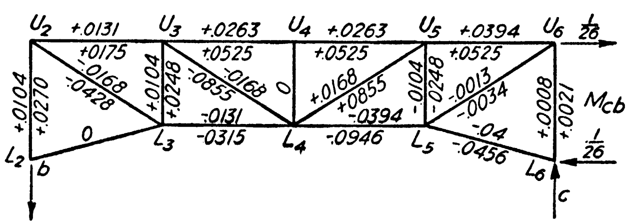


FIG. 111*c*.

The distribution factors at supports b and c are proportional to the coefficients, or

For end spans ba or cd

$$r = \frac{114}{114 + 68} = 0.626$$

For center span

$$r = \frac{68}{182} = 0.374$$

The fixed-end moments for a unit load at U_4 will be calculated from equations 86a and b. The values of $\frac{S'L}{A}$ are recorded on the truss diagram in Fig. 111d, and from these quantities and the previous values of u_b and u_c the end rotations are found to be

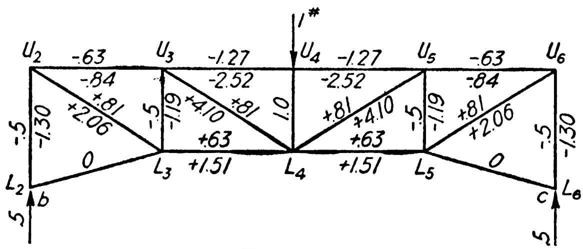


FIG. 111d.

$$\sum \frac{S'u_bL}{A} = 0.427 \qquad \sum \frac{S'u_cL}{A} = -0.427$$

Equations 86a and 86b give

$$\theta_b = 0.427 + M_{Fbc}(0.0155) + M_{Fcb}(-0.00343) = 0$$

$$\theta_c = -0.427 + M_{Fbc}(-0.00343) + M_{Fcb}(0.0155) = 0$$

from which

$$M_{Fbc} = -16.15 \text{ ft-lb} \qquad M_{Fcb} = +16.15 \text{ ft-lb.}$$

The distribution of the fixed-end moments in Fig. 112a follows the same numerical operations as for a continuous beam. The final moment over the supports is 11.02 ft-lb, which makes the reaction R_1 equal to

$$R_1 = \frac{11.02}{48} = 0.229$$

which checks the value that was obtained in Example 35.

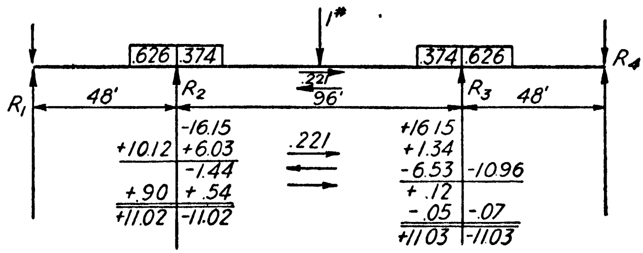


FIG. 112a.

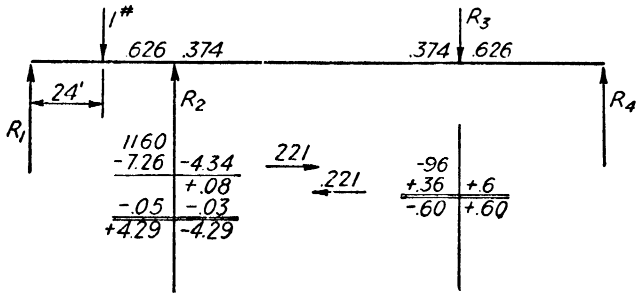


FIG. 112b.

61. Use of an Equivalent Beam. The deformation and stress in the chord member of a truss are similar to those in the flange of a beam. Some differences exist, however, as the stress in a truss member is always constant across a panel whereas the flange stress in a beam ordinarily varies throughout the span. Furthermore, the deformation in the truss diagonals is usually more than the shearing deformation in a beam so that the change in length of the diagonals reduces the carry-over and stiffness factors of a truss more than the shearing deformation does in a beam. In Chapter III it was shown that the end moments in a continuous beam are not appreciably affected by the shearing deformation even though the carry-over and CK values are changed considerably. For this reason the end moments in a continuous truss are approximately the same as those in a continuous beam whose flanges give about the same relative stiffness in the various spans as do the chord members of the truss. In general, the substitution of an equivalent continuous beam for a continuous truss is a logical procedure for making a preliminary design, and usually such a preliminary design will require only slight, if any, modification.

The term *equivalent beam* is a descriptive term only as there is probably no beam that is exactly equivalent in deformation to a truss, or, at least, it is not practical to define such a beam in mathematical terms.

Consequently, the substitution of a continuous beam for a continuous truss is an approximation, but, as already stated, it is a logical procedure because of the similarity in the deformations of the two structures. In general, if a beam of constant width and of the same depth as the truss is used for a preliminary design the moments in such an equivalent beam are remarkably close to those obtained for the actual truss. After one or two values are checked in the final design the designer will usually be satisfied that no changes are necessary.

The differences between the fixed-end moments, distribution factors, and carry-over factors of the truss and those of the equivalent beam, although of considerable magnitude, will seldom need any consideration, as their effect upon the final moments is relatively small. The use of an equivalent beam transforms the problem into the analysis of a continuous beam with variable moment of inertia—a problem that has already been discussed in detail in Chapter VI.

Example 37. The end moments in the continuous truss that was used in Examples 35 and 36 will be calculated by means of an equivalent beam. The depth of the beam will be taken as the same as that of the truss, and the moment of inertia is assumed to vary as the cube of the depth. The coefficients C_1, C_2, and C_3 and the fixed-end moments for such a beam can be selected from curves or calculated by the methods previously explained. In this problem the values are taken from the P.C.A. diagrams in the Appendix. The coefficients for the beams are

End spans

$$\frac{\min d}{\max d} = \frac{19}{26} = 0.73 \qquad a = 0.5$$

$$C_1 = 4.4 \qquad C_2 = 3.0 \qquad C_3 = 7.0$$

$$C_3' = 7.0 - \frac{3.0^2}{4.4} = 5.0$$

$$C_3'K = (5.0)\frac{I_0}{L} = (5.0)\left(\frac{I_0}{48}\right) = 0.104I_0$$

Center span

$$\frac{\min d}{\max d} = 0.73 \qquad a = 0.25$$

$$C_1 = C_3 = 6.0 \qquad \frac{C_2}{C_1} = 0.60$$

$$C_1K = (6.0)\left(\frac{I_0}{L}\right) = (6.0)\frac{I_0}{96} = 0.063I_0$$

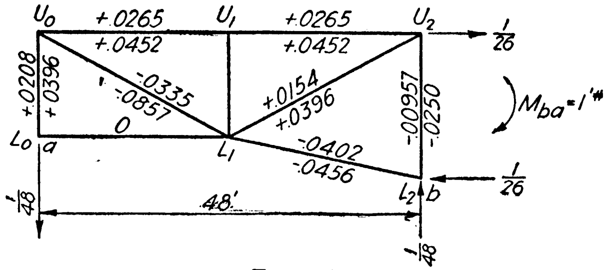


FIG. 111a.

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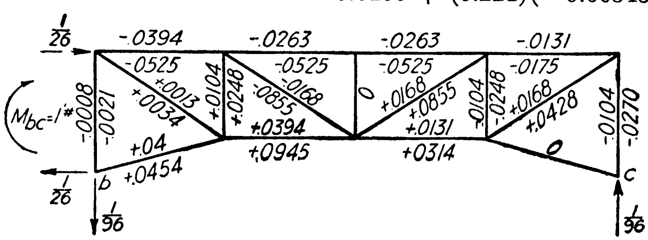

FIG. 111b.

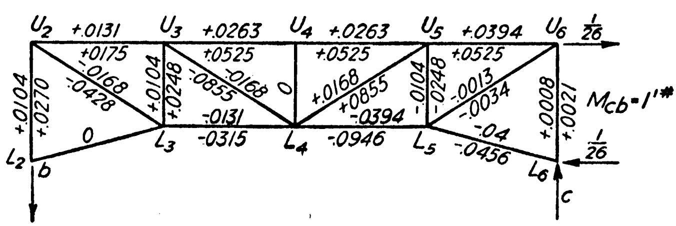

FIG. 111c.

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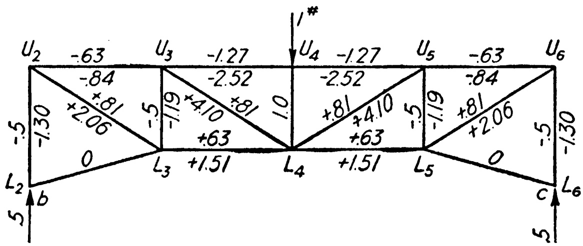

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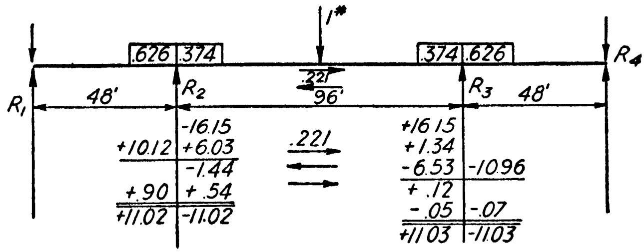
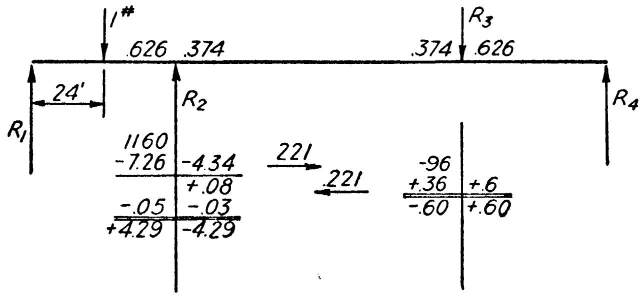


FIG. 112a.



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The differences between the fixed-end moments, distribution factors, and carry-over factors of the truss and those of the equivalent beam, although of considerable magnitude, will seldom need any consideration, as their effect upon the final moments is relatively small. The use of an equivalent beam transforms the problem into the analysis of a continuous beam with variable moment of inertia—a problem that has already been discussed in detail in Chapter VI.

Example 37. The end moments in the continuous truss that was used in Examples 35 and 36 will be calculated by means of an equivalent beam. The depth of the beam will be taken as the same as that of the truss, and the moment of inertia is assumed to vary as the cube of the depth. The coefficients C_1, C_2, and C_3 and the fixed-end moments for such a beam can be selected from curves or calculated by the methods previously explained. In this problem the values are taken from the P.C.A. diagrams in the Appendix. The coefficients for the beams are

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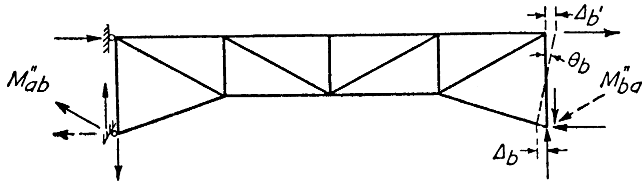

FIG. 110*c*.

59. Distribution and Carry-Over Factors. The determination of the end forces is best illustrated by calculating the end couple M'_{ba}, Fig. 110*b*, that is required to hold joints *b* and *b'* without horizontal translation when some end couple, M'_{ab}, is applied at end *a*. The total strain energy U in the truss for this force system is equal to

$$U = \sum \frac{(M'_{ab}u_a + M'_{ba}u_b)^2 L}{2AE} \tag{75}$$

in which u_a equals stress for $M'_{ab} = 1, \quad M'_{ba} = 0.$

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The end rotations θ_a and θ_b, as given by Castigliano's theorem, are

$$\theta_a = \frac{\partial U}{\partial M'_{ab}} = \sum \frac{(M'_{ab}u_a + M'_{ba}u_b)u_a L}{AE} \tag{76a}$$

$$\theta_b = \frac{\partial U}{\partial M'_{ba}} = \sum \frac{(M'_{ab}u_a + M'_{ba}u_b)u_b L}{AE} \tag{76b}$$

When θ_b is equal to zero, equation 76*b* gives

$$M'_{ba} = -\frac{\sum \frac{u_a u_b L}{A}}{\sum \frac{u_b^2 L}{A}} M'_{ab} = C_{ab} M'_{ab} \tag{77}$$

where

$$C_{ab} = -\frac{\sum \frac{u_a u_b L}{A}}{\sum \frac{u_b^2 L}{A}} \tag{78}$$

is the carry-over factor from a to b. As the term $\sum \frac{u_a u_b L}{A}$ is negative the value of C_{ab} is positive.

If the value of M'_{ba} from equation 77 is now substituted in equation 76a the expression for θ_a becomes

$$\theta_a = \sum \frac{(M'_{ab}u_a + C_{ab}M'_{ab}u_b)u_aL}{AE} \tag{79a}$$

from which

$$M'_{ab} = \frac{E\theta_a}{\sum \frac{u_a^2 L}{A} + C_{ab} \sum \frac{u_a u_b L}{A}} = C_a E \theta_a \tag{79b}$$

where

$$C_a = \frac{1}{\sum \frac{u_a^2 L}{A} + C_{ab} \sum \frac{u_a u_b L}{A}} \tag{79c}$$

is the stiffness factor and corresponds to C_1K_0 or C_3K_0 for a beam.

The end couple M''_{ab}, Fig. 110c, that will hold joints a and a' fixed when a couple M''_{ba} is applied at end b, is calculated in a similar manner. The necessary equations are obtained by interchanging the subscripts a and b in equations 77, 78, and 79, or

$$M''_{ab} = C_{ba}M''_{ba} \tag{80a}$$

in which

$$C_{ba} = -\frac{\sum \frac{u_a u_b L}{A}}{\sum \frac{u_a^2 L}{A}} \tag{80b}$$

is the carry-over factor from b to a. Also,

$$M''_{ba} = C_b E \theta_b \tag{81a}$$

where

$$C_b = \frac{1}{\sum \frac{u_b^2 L}{A} + C_{ba} \sum \frac{u_b u_a L}{A}} \tag{81b}$$

After the coefficients C_a and C_b have been computed for each span the distribution factors r can be determined, as

$$r_{ab} = \frac{C_a}{\Sigma C_a} \quad \text{and} \quad r_{ba} = \frac{C_b}{\Sigma C_b} \tag{82}$$

or the ratio of the coefficient of one span to the sum of the coefficients for both spans.

Equation 79c can be used when the truss is simply supported at b by making u_b equal to zero, or

$$C_a = \frac{1}{\sum \frac{u_a^2 L}{A}} \tag{83a}$$

When the truss is simply supported at a, the coefficient C_b becomes

$$C_b = \frac{1}{\sum \frac{u_b^2 L}{A}} \tag{83b}$$

60. Fixed-End Couples. Either the algebraic or preferably the semi-graphic method can be used to calculate the end couples required to hold points a, a', b, b' without translation when any load is applied on the span. The stress in any member is equal to

$$S = S' + M_{Fab}u_a + M_{Fba}u_b \tag{84}$$

where S' = stress due to applied load on a simply supported span.
u_a, u_b = stress due to unit end couples at a and b, respectively.

The total strain energy in the truss is

$$U = \sum \frac{S^2 L}{2AE} = \sum \frac{(S' + M_{Fab}u_a + M_{Fba}u_b)^2 L}{2AE} \tag{85}$$

and the end rotations θ_a and θ_b equal

$$\theta_a = \frac{\partial U}{\partial M_{Fab}}$$

$$= \frac{1}{E}\left\{\sum \frac{S'u_a L}{A} + M_{Fab}\sum \frac{u_a^2 L}{A} + M_{Fba}\sum \frac{u_a u_b L}{A}\right\} = 0 \tag{86a}$$

$$\theta_b = \frac{\partial U}{\partial M_{Fba}}$$

$$= \frac{1}{E}\left\{\sum \frac{S'u_b L}{A} + M_{Fab}\sum \frac{u_a u_b L}{A} + M_{Fba}\sum \frac{u_b^2 L}{A}\right\} = 0 \tag{86b}$$

The fixed-end couples can be obtained by solving equations 86a and 86b simultaneously. These equations are similar to equations 69a and b except that the summation is for one span only instead of the entire structure.

Influence diagrams for the fixed-end moments or couples in a continuous truss are best obtained directly from an elastic curve of the structure by applying the Müller-Breslau principle as in Article 51 for the fixed-end moments in continuous beams. The procedure is similar to that followed in other problems that have been solved by means of the reciprocal theorem. If some end couple M'_{ab} is applied at end a, Fig. 110b, the couple M'_{ba} required to hold end b is, from equation 77,

$$M'_{ba} = C_{ab}M'_{ab}$$

The stresses S and $\frac{SL}{A}$, the change in length times E, must be determined for any assumed value of M'_{ab}. If M'_{ab} is taken equal to unity

$$S = u_a + C_{ab}u_b$$

$$\frac{SL}{A} = \frac{u_aL}{A} + C_{ab}\frac{u_bL}{A}$$

The displacements y and rotation θ_a are given directly by a Williot diagram drawn for the $\frac{SL}{A}$ values if point b is selected as a fixed point and bb' as a fixed axis. The ordinates to the influence diagram for M_{Fab}, as in other problems, are equal to

$$M_{Fab} = P\frac{y}{\theta_a} = Ph_a\left(\frac{y}{\Delta_a + \Delta'_a}\right) \tag{87}$$

Example 36. The end moments and reactions of the continuous truss of Example 35, Fig. 105, will be determined for a unit load at point U_4 by the moment-distribution method.

The stiffness coefficient C_b for the end spans is given by equation 83b as the spans are simply supported at L_0 and L_8. The values of u_b and $\frac{u_bL}{A}$ for a unit end moment applied to the end span are recorded in Fig. 111a. From these values, the coefficient C_{b1} is equal to

$$C_{b1} = \frac{1}{\sum \frac{u_b^2L}{A}} = \frac{1}{0.00877} = 114$$

Substituting these quantities in equation 73 gives the value of R_2' as

$$R_2' = -\left(\frac{347.5}{222.7}\right) R_1' = -1.56R_1'$$

For an applied load R_1' equal to unity the reactions that will prevent any vertical displacement at L_2, L_6, and L_8 and also satisfy the equilibrium requirements are

$$R_2' = -1.56 \qquad R_3' = 0.677 \qquad R_4' = -0.118$$

The stresses S in the truss members for R_1' equal to unity are therefore equal to

$$S = u_1 - 1.56u_2$$

and the change in length of each member (times E) is

$$\frac{SL}{A} = \frac{u_1 L}{A} - 1.56\,\frac{u_2 L}{A}$$

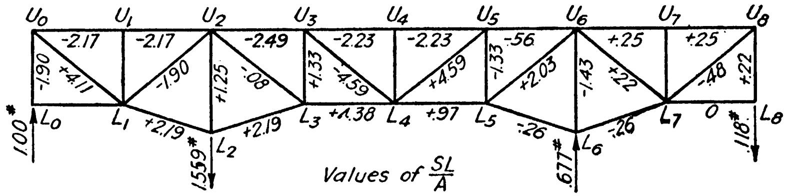
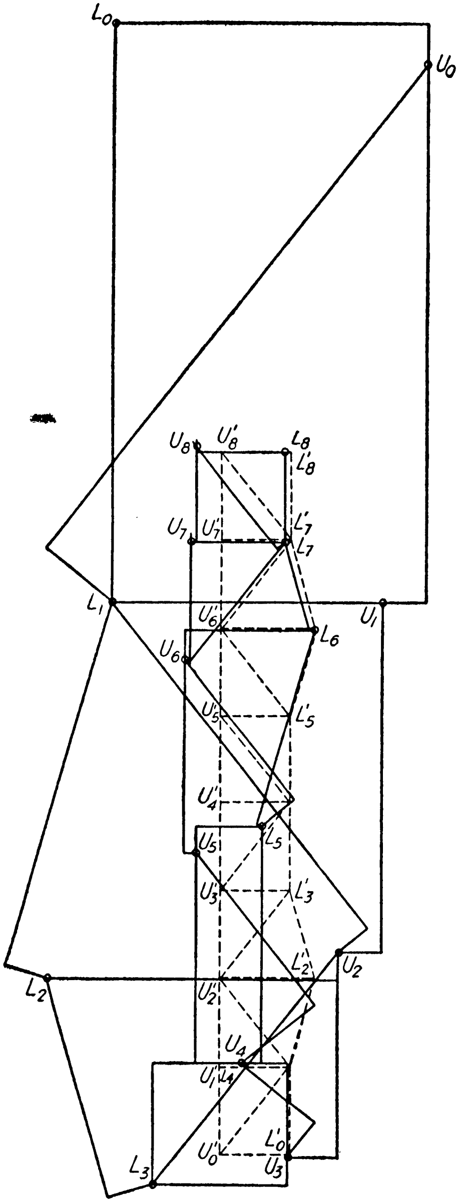


FIG. 106c.

The numerical values of $\frac{SL}{A}$ as recorded on the truss diagram in Fig. 106c were used to draw the Williot diagram in Fig. 107. In this diagram L_4 was selected as the fixed point and member U_4L_4 as a fixed axis. The construction of the displacement diagram which follows the procedure explained in Article 16 should be executed accurately on a large drawing board with good equipment. The position of each point should be back-checked as soon as it is determined to avoid extending any error to succeeding points.

After the Williot diagram is completed it is apparent that U_4L_4 is not a fixed axis, as such an assumption gives a relative vertical displacement of points L_2, L_6, and L_8. Consequently the structure is rotated about L_6 as a fixed point until the displacement at L_2 is zero. The displacements due to the rotation are given by the Mohr rotation diagram, which is indicated by broken lines. As the displacement of L_8 must also be zero a check on the accuracy of the drawing work is obtained. The vertical displacement of any point is equal to the



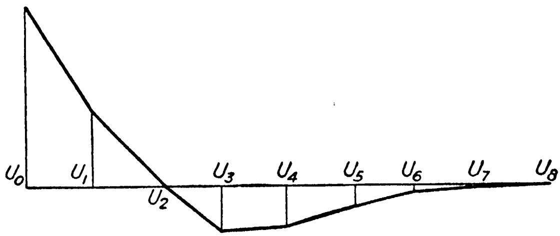
Displacement Diagram
For A Continuous Truss
Scale 1" = 4 units

FIG. 107.

vertical distance from the point on the Mohr diagram to the corresponding point on the Williot diagram. The vertical movement of L_0 is equal to the vertical component of the vector $L_0'L_0$ or 55.0 units, and the vertical displacement of U_1 is 22.5 units. The value of R_1 in Fig. 105 for a unit load at U_1 is, therefore,

$$R_1 = \frac{y}{\Delta_a} = \frac{22.5}{55.0} = 0.409$$

The vertical displacement of all upper panel points and the corresponding ordinates to the influence diagram are tabulated in Table 8. The influence diagram in Fig. 108 is drawn from these values.



Influence Diagram for R_1

FIG. 108.

TABLE 8

POINT	VERTICAL DISPLACEMENT	VALUE OF R_1 $\frac{y}{55.0}$
U_0	+53.1	+0.965
U_1	+22.5	+0.409
U_2	+1.2	+0.022
U_3	−12.9	−0.235
U_4	−12.6	−0.229
U_5	−6.6	−0.120
U_6	−1.4	−0.025
U_7	−0.1	−0.002
U_8	+0.2	+0.004

A check on the above values is obtained by computing the displacement of L_0 algebraically from equation 74, which gives

$$\Delta_1 = \frac{1}{E} \sum u_1 \frac{SL}{A} = \frac{55.0}{E}$$

which is identical with the value obtained from the Williot diagram.

58. End Forces and Couples. The forces acting upon any span of a continuous truss, such as span *ab*, Fig. 109, are analogous to those in a continuous beam inasmuch as they consist of the applied loading, end shears, and end couples. Moreover, as in the analysis of continuous beams, any span *ab*, Fig. 109, can first be assumed as fixed at the ends

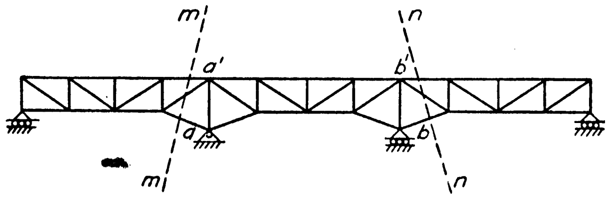


Fig. 109.

and these fixed-end forces afterwards corrected so as to provide the necessary equilibrium and strain conditions between continuous spans. The horizontal components or end couples at a and a', also b and b', must balance, and the horizontal displacement of these points must be the same for both adjacent spans.

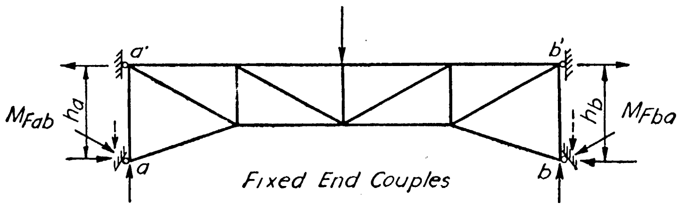


Fig. 110*a*.

To illustrate the above statements graphically, the fixed-end forces in Fig. 110*a* must be corrected by adding the end forces in Figs. 110*b* and *c* so as to make the end rotations

$$\theta_a = \frac{\Delta_a + \Delta_{a'}}{h_a} \quad \text{and} \quad \theta_b = \frac{\Delta_b + \Delta_{b'}}{h_b}$$

equal for both spans. These end forces or couples can be expressed in terms of the rotation θ_a and θ_b, as was done for the end couples applied

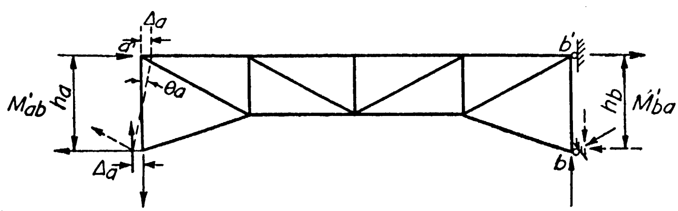


Fig. 110*b*.

In equation 66*c*, u' is the stress in the member of the simply supported truss for P equal to unity, and with R_1 and R_2 regarded as part of the applied forces. Equation 66*c* can also be written so as to regard R_1 and R_2 as functions of P, that is

$$\Delta_m = \frac{\partial U}{\partial P} = \sum \frac{S \dfrac{\partial S}{\partial P} L}{AE} = \sum \frac{SuL}{AE} \tag{66d}$$

where u is the resultant stress S in the member of the continuous truss for P equal to unity. Although u' is not equal to u, the summations of the terms in equations 66*c* and 66*d* are the same as it is immaterial whether the truss is regarded as a simply supported truss subjected to applied loads of P, R_1, and R_2 or as a continuous truss subjected to the load P.

If the displacement is desired at any point at which a load is not applied, an auxiliary or virtual force P_x, applied at the point, must be included in the forces acting on the structures.

For such a combined force system, the total stress S in any member is

$$S = S' + R_1 u_1 + R_2 u_2 + S_x \tag{67}$$

in which S_x is the stress due to P_x, the auxiliary force.

The displacement Δ at the point where P_x is applied is

$$\Delta_x = \frac{\partial U}{\partial P_x} = \sum \frac{(S' + R_1 u_1 + R_2 u_2 + S_x) u_x L}{AE}$$

but, since the value of S_x is actually zero,

$$\Delta_x = \sum \frac{S u_x L}{AE} \tag{68}$$

in which S is the stress due to the actual load and u_x is the stress due to an auxiliary load of unity applied at point x on the *simply supported* structure.

All the above displacements can be obtained by combining the separate displacements for each force system in Fig. 103. For example, the displacement Δ_m is equal to the sum

$$\Delta_m = \Delta'_m + \Delta''_m + \Delta'''_m = \sum \frac{S'u'L}{AE} + \sum \frac{R_1 u_1 u' L}{AE} + \sum \frac{R_2 u_2 u' L}{AE}$$

which is identical with equation 66*c*.

56. Redundant Reactions. The reactions of continuous trusses are frequently calculated from strain equations obtained by assuming that

the supports undergo no vertical displacements. In Fig. 102, if the displacements at R_1 and R_2 are equal to zero, equations 66a and 66b can be written in the form

$$\Delta_1 = \frac{1}{E}\left\{\sum \frac{S'u_1L}{A} + R_1\sum \frac{u_1^2L}{A} + R_2\sum \frac{u_1u_2L}{A}\right\} = 0 \quad (69a)$$

$$\Delta_2 = \frac{1}{E}\left\{\sum \frac{S'u_2L}{A} + R_1\sum \frac{u_1u_2L}{A} + R_2\sum \frac{u_2^2L}{A}\right\} = 0 \quad (69b)$$

If the constant terms in the above equations are represented by

$$C_1 = \sum \frac{u_1^2L}{A} \qquad C_{12} = \sum \frac{u_1u_2L}{A} \qquad C_2 = \sum \frac{u_2^2L}{A}$$

$$C' = \sum \frac{S'u_1L}{A} \qquad C'' = \sum \frac{S'u_2L}{A}$$

the reactions R_1 and R_2 are given by the following expressions:

$$R_1 = \frac{C''C_{12} - C'C_2}{C_1C_2 - C_{12}^2} \quad (70a)$$

$$R_2 = \frac{C'C_{12} - C''C_1}{C_1C_2 - C_{12}^2} \quad (70b)$$

Similar equations can be established for a continuous truss of any number of spans. For a four-span continuous truss, another redundant reaction, say R_3, must be used, and the following additional constant terms computed:

$$C_3 = \sum \frac{u_3^2L}{A} \qquad C_{13} = \sum \frac{u_1u_3L}{A} \qquad C_{23} = \sum \frac{u_2u_3L}{A}$$

$$C''' = \sum \frac{S'u_3L}{A}$$

For a continuous truss of four spans that has no vertical displacements at the supports the following equations can be written

$$\Delta_1 = C_1R_1 + C_{12}R_2 + C_{13}R_3 + C' = 0 \quad (71a)$$

$$\Delta_2 = C_{12}R_1 + C_2R_2 + C_{23}R_3 + C'' = 0 \quad (71b)$$

$$\Delta_3 = C_{13}R_1 + C_{23}R_2 + C_3R_3 + C''' = 0 \quad (71c)$$

from which the values of R_1, R_2, and R_3 can be computed.

57. Influence Diagrams for Reactions. A semi-graphical method for obtaining the influence diagram for any redundant reaction of a continuous truss is often superior to the algebraic procedure just explained. The reciprocal theorem forms the basis of the method, while the Williot and Mohr displacement diagrams provide a convenient graphical solution for calculating the necessary deflections. If some force R_1' is applied to a truss as in Fig. 104, and the elastic curve of the structure drawn, then it will be found from the reciprocal theorem that the reaction R_1, Fig. 102, for any applied load P is equal to

$$R_1 = P\frac{y}{\Delta_1} \tag{72}$$

Before the values of y and Δ_1 can be obtained, the reactions and stresses due to R_1', Fig. 104, must be known. This part of the solution

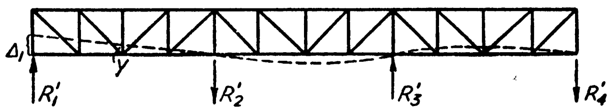


FIG. 104.

is best made by the algebraic methods, as the calculations are not difficult and also they provide a check upon the graphical solution. From equation 66*b*, with S' equal to zero, the relation between R_1' and R_2' to make Δ_2 equal to zero is

$$R_2' = -R_1'\frac{\sum\frac{u_1u_2L}{A}}{\sum\frac{u_2^2L}{A}} \tag{73}$$

After equation 73 is solved, the stresses S in the members and the change in length of each member, $\frac{SL}{AE}$, are computed and the displacement diagrams drawn. The length L can be taken in foot units and $\frac{1}{E}$ omitted for convenience as only the ratios of displacements are used in the solution. A check upon all displacements can be obtained by computing Δ_1 from the equation

$$\Delta_1 = \frac{1}{E}\sum u_1\frac{SL}{A} \tag{74}$$

This check on the results of the graphical solution can be quickly made as the terms in equation 74 are already known.

Example 35. The ordinates to the influence diagram for the reaction R_1 of the three-span continuous truss in Fig. 105 will be calculated by the semi-graphical method. The lengths and areas of the members

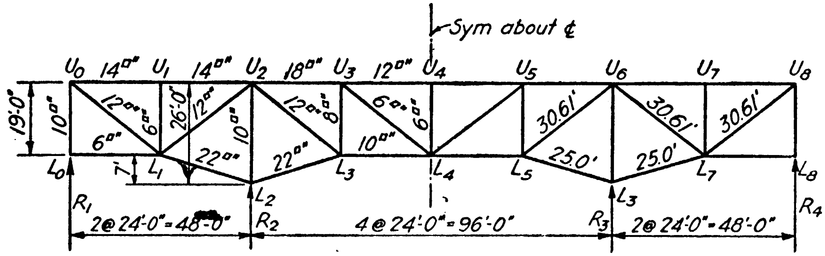


FIG. 105.

are recorded on the truss diagram. Values of u_1 and $\frac{u_1L}{A}$ for R_1 equal to unity and of u_2 and $\frac{u_2L}{A}$ for R_2 equal to unity are given in Figs. 106*a* and 106*b*, respectively. By means of equation 73, the reaction R_2'

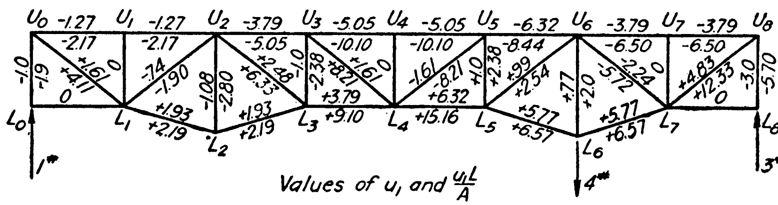


FIG. 106*a*.

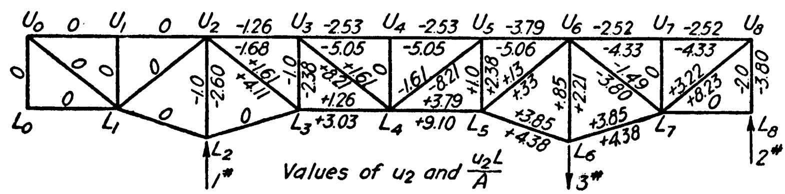


FIG. 106*b*.

is obtained directly from the u_1, u_2, and $\frac{u_2L}{A}$ quantities for any applied force R_1'. The summations of these terms for all members of the truss are

$$\sum \frac{u_1u_2L}{A} = 347.5 \qquad \sum \frac{u_2^2L}{A} = 222.7$$

on pages 327 and 328, the following coefficients are obtained for the member *ab*.

$$C_1 = 5.63 \qquad C_2 = 5.95 \qquad C_3 = 26.50$$

$$\text{Carry-over factor from } a \text{ to } b = \frac{5.95}{26.50} = 0.22$$

$$\text{Carry-over factor from } b \text{ to } a = \frac{5.95}{5.63} = 1.06$$

Distribution factors at b

MEMBER	C	K	CK	r
ab	5.63	$\frac{I_0}{40}$	$0.141I_0$	0.16
bc	3.00	$\frac{I_0}{4}$	$0.750I_0$	0.84
			$\Sigma CK_0 = 0.891I_0$	1.00

The bending moments in the columns for a wind load of 0.5 kip per foot will be calculated by the procedure that has been explained in Example 39. The values for the fixed-end moments which are calculated directly from the curves on page 330 are

$$M_{Fab} = -0.111wL^2 = (-0.111)(0.5)(40)^2 = -88.8 \text{ ft-kips}$$
$$M_{Fba} = +0.0518wL^2 = 41.5 \text{ ft-kips}$$
$$M_{Fbc} = -0.125wL^2 = (-0.125)(0.5)(4)^2 = -1.0 \text{ ft-kip}$$

After the unbalanced moment at b, 40.5 ft-kips, is distributed by means of the distribution factors recorded above, the final moments become, (see Fig. 120*a*)

$$M_{ab} = -88.8 + (1.06)(0.16)(-40.5) = -95.7 \text{ ft-kips}$$
$$M_{ba} = -M_{bc} = 41.5 + (0.16)(-40.5) = 35.0 \text{ ft-kips}$$
$$H_a = 11.52 \text{ kips} \leftarrow \quad H_b = 18.23 \text{ kips} \leftarrow \quad H_c = 7.75 \text{ kips} \rightarrow$$

The end moments due to a horizontal crane load of 8 kips acting to the right can be quickly calculated once the fixed-end moments are determined. From the coefficients on page 329 the values of the fixed-end moments can be obtained directly.

$$M_{Fab} = -0.121PL = (-0.121)(8)(40) = -38.7 \text{ ft-kips}$$
$$M_{Fba} = 0.0994PL = +31.8 \text{ ft-kips}$$
$$M_{Fbc} = 0$$

After the necessary distribution of the unbalanced moment of 31.8 ft-kips at b, the final moments are

$$M_{ab} = -38.7 - 5.4 = -44.1 \text{ ft-kips}$$

$$M_{ba} = -M_{bc} = 31.8 - 5.1 = 26.7 \text{ ft-kips}$$

$$H_a = 2.83 \text{ kips} \leftarrow \quad H_b = 11.85 \text{ kips} \leftarrow \quad H_c = 6.68 \text{ kips} \rightarrow$$

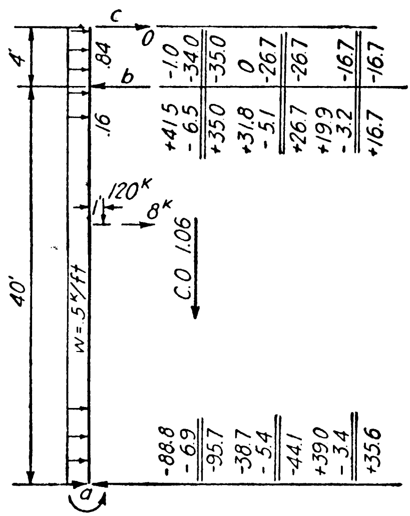


FIG. 120a.

The fixed-end moments for an eccentric vertical load of 120 kips which is equivalent to an axial load of 120 kips and a couple of 120 ft-kips are also determined directly from the curves on page 331. These values are

$$M_{Fab} = 0.325M_x = +39.0 \text{ ft-kips}$$

$$M_{Fba} = 0.166M_x = +19.9 \text{ ft-kips}$$

which, after the distribution of the unbalanced moment at b, become

$$M_{ab} = 39.0 - 3.4 = 35.6 \text{ ft-kips}$$

$$M_{ba} = 19.9 - 3.2 = 16.7 \text{ ft-kips}$$

$$H_a = 4.31 \text{ kips} \rightarrow \quad H_b = 0.13 \text{ kips} \leftarrow \quad H_c = 4.18 \text{ kips} \leftarrow$$

The moments caused by the wind load and horizontal crane load will be corrected for sidesway by the customary procedure of applying

forces at points b and c that are equal and opposite to the restraining forces previously computed. Let points b and c both move 40 units to the right, and assume K_0 equal to unity. The relative values of the fixed-end moments for this condition are

$$M_{Fab} = (C_2 + C_3)K_0\frac{\Delta}{L} = -(5.95 + 26.50)(1)(\tfrac{40}{40}) = -32.45 \text{ ft-kips}$$

$$M_{Fba} = (C_1 + C_2)K_0\frac{\Delta}{L} = -(5.63 + 5.95)(1)(\tfrac{40}{40}) = -11.58 \text{ ft-kips}$$

The final moments that are obtained by rotating joint b to a condition of equilibrium (see Fig. 120b) are

$$M_{ab} = -32.45 + (1.06)(1.85) = -30.48 \text{ ft-kips}$$

$$M_{ba} = -11.58 + 1.85 = -9.73 \text{ ft-kips}$$

$$H_a = 1.01 \text{ kips} \leftarrow \quad H_b = 3.44 \text{ kips} \rightarrow \quad H_c = 2.43 \text{ kips} \leftarrow$$

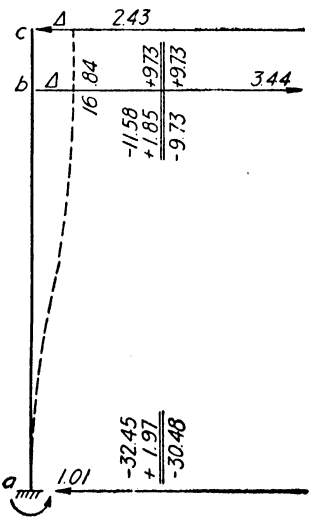


FIG. 120b.

The final moments in the columns for the separate loads can now be determined by adding the corrections due to sidesway to the original values. Thus, for the wind load the correction for sidesway is obtained by multiplying the moments in Fig. 120b by the ratio $\dfrac{5.24}{1.01} = 5.18$.

Therefore, the final values of the column moments due to the wind force are

Column A

$$M_{ab} = -95.7 + (5.18)(-30.48) = -254.2 \text{ ft-kips}$$

$$M_{ba} = +35.0 + (5.18)(-9.73) = -15.5 \text{ ft-kips}$$

$$H_a = 11.52 + 5.24 = 16.76 \text{ kips}$$

Column B

$$M_{a'b'} = -158.5 \text{ ft-kips}$$

$$M_{b'a'} = -50.5 \text{ ft-kips}$$

$$H_{a'} = 5.24 \text{ kips}$$

The final moments in the columns due to the horizontal crane load are determined by correcting the original values for an additional shear of $\frac{8 - 2.83}{6}$, or 0.86 kip, acting on each column. The total restraining force of 5.17 kips is divided between six columns rather than two as it is assumed that the horizontal bracing will distribute the load over three bents. The corrections of the end moments for sidesway are therefore obtained by multiplying the moments in Fig. 120*b* by $\frac{0.86}{1.01}$ or .852. The final moments in the columns due to the horizontal crane load are

Column A

$$M_{ab} = -44.1 + (0.852)(-30.48) = -70.1 \text{ ft-kips}$$

$$M_{ba} = 26.7 + (0.852)(-9.73) = 18.4 \text{ ft-kips}$$

$$H_a = 2.83 + 0.86 = 3.69 \text{ kips} \leftarrow$$

Column B

$$M_{a'b'} = -26.0 \text{ ft-kips}$$

$$M_{b'a'} = -8.3 \text{ ft-kips}$$

$$H_{a'} = 0.86 \text{ kip} \leftarrow$$

The moments resulting from the application of the vertical crane loads can also be corrected for sidesway if necessary. In this problem, as the vertical crane loads are assumed to be symmetrical, no correction for sidesway is required.

Maximum combined moments, assuming that the horizontal crane load can be applied to either column in either direction are recorded below. These values, which are presented to illustrate the numerical calculations, are not intended to represent all possible loading conditions.

Column A

$$M_{ab} = \text{Wind load} + \text{Horizontal crane load} + \text{Vertical crane load}$$
$$= -254.2 - 70.0 + 35.6 = -288.6 \text{ ft-kips}$$
$$M_{ba} = \text{Horizontal crane load} + \text{Vertical load} = 18.4 + 16.7 = 35.1 \text{ ft-kips}$$

Column B

$$M_{a'b'} = -158.1 - 70.0 - 35.6 = -263.7 \text{ ft-kips}$$
$$M_{b'a'} = \text{Wind load} + \text{Horizontal crane load} + \text{Vertical crane load}$$
$$= -50.5 - 18.4 - 16.7 = -85.6 \text{ ft-kips}$$

68. Two-Hinged Arch Trusses. Two-hinged arch trusses in which the horizontal reaction is resisted either by the abutments or by a tie

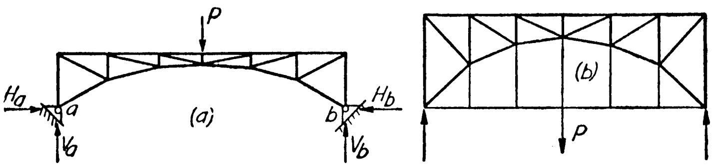


FIG. 121.

(Figs. 121a, b) are readily solved by Castigliano's theorem. The stress S in any member is equal to

$$S = S' + Hu$$

in which S' = stress in any member of a simply supported truss with H removed (Fig. 122a).

u = stress in any member of a simply supported truss for H equal to unity (Fig. 122b).

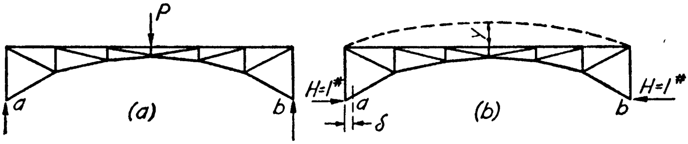


FIG. 122.

The total strain energy U equals

$$U = \sum \frac{S^2 L}{2AE} = \sum \frac{(S' + Hu)^2 L}{2AE}$$

Then

$$\frac{dU}{dH} = \sum \frac{(S' + Hu)uL}{AE} = 0$$

from which

$$H = -\frac{\sum \frac{S'uL}{A}}{\sum \frac{u^2 L}{A}} \tag{100}$$

When the arch has a horizontal tie between supports the denominator of equation 100 must include the quantity $\frac{L}{A}$ for the tie but the numerator is not affected. The theoretical value of H will therefore be slightly less than when the arch is rigidly supported by abutments. Actually a displacement of the abutments may reduce H more than the elongation of the tie bar.

Since

$$\frac{dU}{dH} = \sum \frac{(S' + Hu)uL}{AE} = \Delta$$

therefore

$$H = -\frac{\sum \frac{S'uL}{AE} - \Delta}{\sum \frac{u^2 L}{AE}} \tag{101}$$

where Δ is the increase in span due to a relative movement of the abutments. It is interesting to note that the elongation of the tie bar increases the denominator in the expression for H, equation 100, while a displacement of the supports decreases the numerator in equation 101. However, the two structures are not easily compared as the two displacements are likely to be quite different quantitatively. The use of a tie bar will usually depend upon foundation conditions as well as other design features.

By drawing a Williot diagram for the arch for any assumed value of H, Fig. 122*b*, an influence diagram for H is obtained directly. From the reciprocal theorem,

$$H\delta - Py = 0$$

or

$$H = P\frac{y}{\delta}$$

therefore only y and δ, which are given directly by a Williot diagram, are required. This method of solution reduces the problem to a statically determinate one.

For preliminary design a trial value of H can be obtained by assuming $\frac{L}{A}$ or $\frac{uL}{A}$ as constant for all members. The trial value of H then becomes

$$H = -\frac{\Sigma S'u}{\Sigma u^2} \tag{102a}$$

or

$$H = -\frac{\Sigma S'}{\Sigma u} \tag{102b}$$

The final analysis with the proper areas should not require many corrections (see Ref. 7·1). Temperature changes will cause stresses in two-hinged arch trusses that are supported on abutments that prevent any horizontal movement. The horizontal reaction H due to any temperature change is equal to

$$H = \pm \frac{\alpha t L}{\sum \frac{u^2 L}{AE}} \tag{103}$$

in which α = coefficient of linear expansion.
t = change in temperature.
L = span length.

A tied arch truss is not affected by temperature change unless the tie bar is subjected to a different temperature from the truss.

69. Continuous Arch Trusses. The structure illustrated in Fig. 123, which is a combination of a continuous truss and a tied arch truss, is

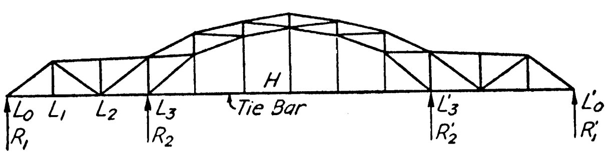


FIG. 123.

readily analyzed by the same methods as for continuous trusses (see Refs. 7·6 and 7·7). The redundant reactions R_1, R_2, and H (or, if the structure is symmetrical about the center line, as it usually is, R_2, R_2', and H) can be calculated from equations 71a, b, and c by replacing one of the vertical reactions by H, the stress in the tie bar. As the structure

must be designed for moving concentrated loads the construction of influence diagrams for the redundant reactions by the semi-graphical method used in example 35 is advisable.

The influence diagram for H is obtained by applying any convenient stress to the tie bar, say 10 kips (Fig. 124), calculating the redundant reactions R_2 and R_2', and then drawing a Williot displacement diagram for

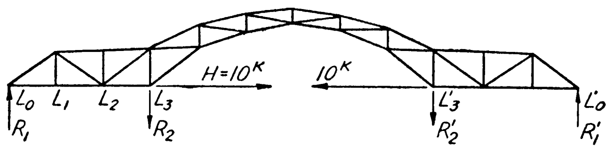


FIG. 124.

the structure. For a symmetrical structure R_2' is equal to R_2, and, therefore,

$$R_2 = -H \frac{\sum \frac{uu_2L}{A}}{\sum \frac{u_2^2L}{A} + \sum \frac{u_2'u_2L}{A}} \tag{104}$$

in which u = stress in any member for H equal to unity with R_2 and R_2' removed.

u_2 = stress in any member for R_2 equal to unity with H and R_2' removed.

u_2' = stress in any member for R_2' equal to unity with H and R_2 removed.

The values of u_2' are antisymmetrical with u_2 for a symmetrical structure.

The stress S in any member is equal to

$$S = Hu + R_2u_2 + R_2'u_2' = Hu + R_2(u_2 + u_2') \tag{105}$$

After the values of $\frac{SL}{A}$ are tabulated a Williot diagram is then drawn for half of the structure. If the diagram is started at the pin L_3 with the chord L_2L_3 as the fixed axis the construction errors will be greatly reduced. A rotation diagram to bring point L_0 back to the required elevation must be constructed next, after which the necessary vertical and horizontal displacements can be scaled from the combined diagrams. The elongation of the tie bar must be added to the horizontal movement between the panel points L_3 and L_3' to obtain the total dis-

placement. For a symmetrical structure the diagram need be drawn for only one half of the truss.

The influence diagrams for R_2 and R_2' are constructed semi-graphically in a similar manner by applying any convenient value of R_2, say 10 kips. The redundant forces R_2' and H must be determined next, after which the stress in each member is calculated and the Williot and Mohr displacement diagrams are drawn. The redundant forces R_2' and H are calculated from the following equations:

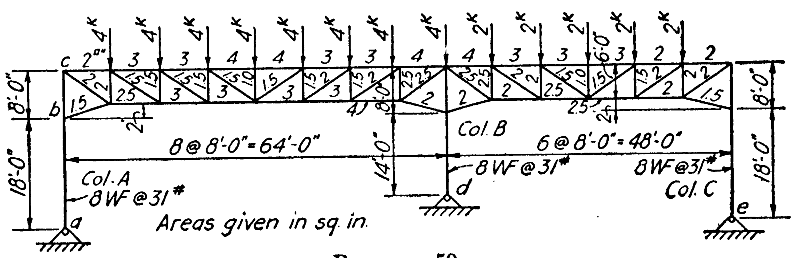
$$H\sum\frac{u^2L}{A} + R_2'\sum\frac{u_2'uL}{A} = -R_2\sum\frac{u_2uL}{A} \tag{106a}$$

$$H\sum\frac{u_2'uL}{A} + R_2'\sum\frac{u_2'^2L}{A} = -R_2\sum\frac{u_2u_2'L}{A} \tag{106b}$$

in which u, u_2, u_2' have the same values as in equation 104. The displacement diagram must be drawn for the entire structure, although the construction can proceed from one point or it can be made in two or more parts.

PROBLEMS

59. Calculate the stresses in all members of the continuous truss shown. Assume that the truss is simply supported at columns A and C and is continuous over column B. Neglect the flexural resistance of the columns. Compare the values of the reactions with those of an equivalent continuous beam.



PROBLEM 59.

60. Construct an influence diagram for the fixed-end moment at the right end of the 64-ft span of the continuous truss in Problem 59 if the left end is simply supported. Make use of the reciprocal theorem and Williot diagram.

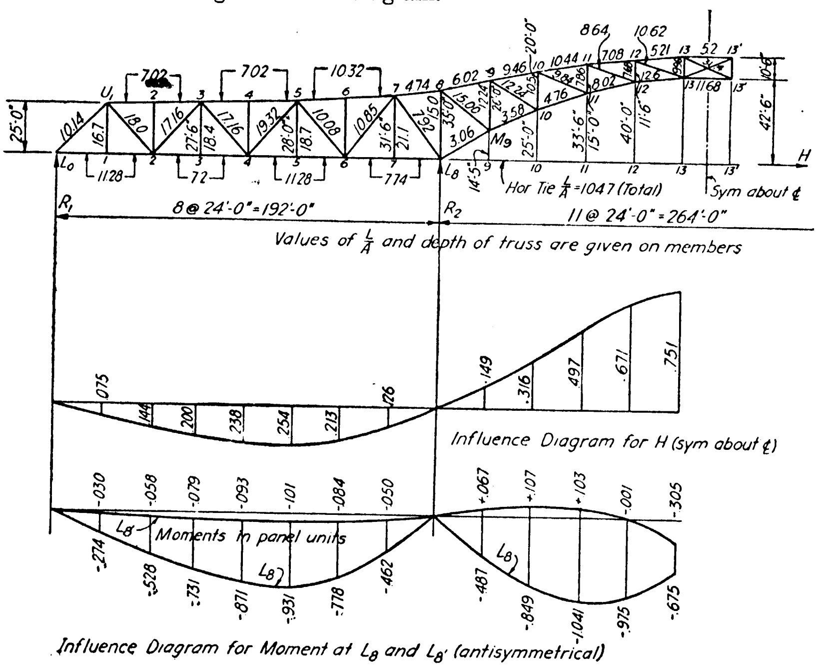
61. Construct an influence diagram for the fixed-end moment at the left end of the 48-ft span in Problem 59 if the right end is simply supported.

62. From the results of Problems 60 and 61, calculate the fixed-end moments and distribution factors for the loads given in Problem 59. Distribute the unbalanced moment, and check with the results obtained in Problem 59.

63. Calculate the reactions at the base of all columns of the bent in Problem 59 for a uniform horizontal wind load of 400 lb per linear foot applied to Column A.

64. Solve Problem 63 for the columns fixed at the base.

65. Construct an influence diagram for the stress in the horizontal tie of the continuous truss arch shown. This bridge was constructed over the Meramec River by the Missouri State Highway Dept. (see Ref. 7·6). Use the semi-graphical method, and check the results with the values given. Values of $\frac{L}{A}$ for all members are given on the diagram.



PROBLEM 65.

66. Construct an influence diagram for the vertical reactions R_2 and R_2' of the continuous truss arch in Problem 65. Compare the moments at the support R_2 with the values given in the diagram. Note that the influence diagrams are antisymmetrical and that, as the moments are expressed in panel units, they must be multiplied by 24 ft.

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7·3 P. C. Hu, "Comparison of Fixed-End Moments, Distribution Factors, and Carry-Over Factors in Trusses and Corresponding Beams," thesis for M.S. degree in engineering, University of Michigan.

7·4 O. T. Voodhuigula, "Analysis of Statically Indeterminate Trussed Structures by Successive Approximations," *Trans. Am. Soc. C. E.*, Vol. 107, 1942.

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7·6 Howard H. Mullins, "Continuous Tied Arch Built in Missouri," *Engg. News-Record*, June 5, 1941, p. 896.

7·7 "A Three-Span Continuous Truss Bridge with the Middle Span a Tied Arch," *Engg. News-Record*, Feb. 25, 1943, p. 42.

CHAPTER VIII

ELASTIC ARCHES, RINGS, AND FRAMES WITH CURVED MEMBERS

70. Two-Hinged Arches. As an arch that is hinged at both supports has one redundant reaction (either horizontal component H), the analysis must begin by the determination of that force. Thus, in Fig. 125a,

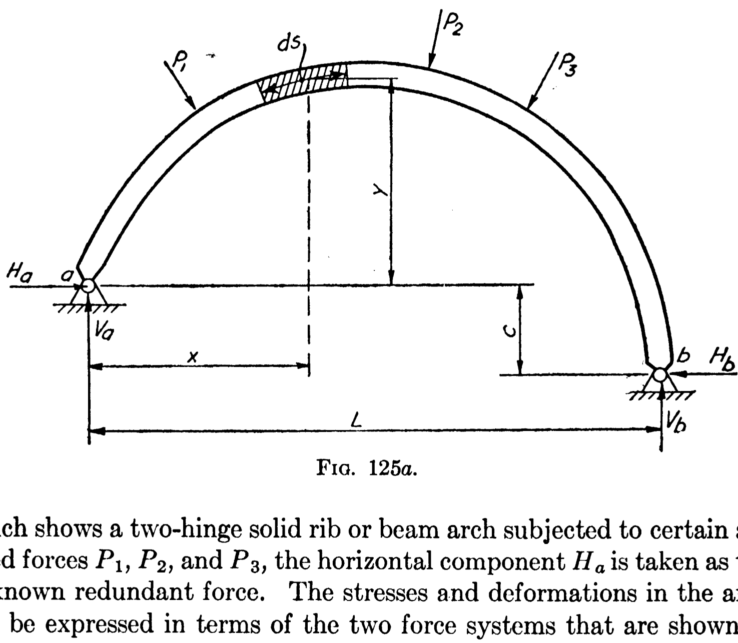


FIG. 125a.

which shows a two-hinge solid rib or beam arch subjected to certain applied forces P_1, P_2, and P_3, the horizontal component H_a is taken as the unknown redundant force. The stresses and deformations in the arch can be expressed in terms of the two force systems that are shown in Figs. 125b and c, and for which the total strain energy (neglecting shear) is

$$U = \int_a^b \frac{(M_s + M')^2 ds}{2EI} + \int_a^b \frac{(N_s + N')^2 ds}{2AE} \tag{107}$$

The relative horizontal displacement between supports a and b is

$$\frac{\partial U}{\partial H_a} = \Delta_a$$

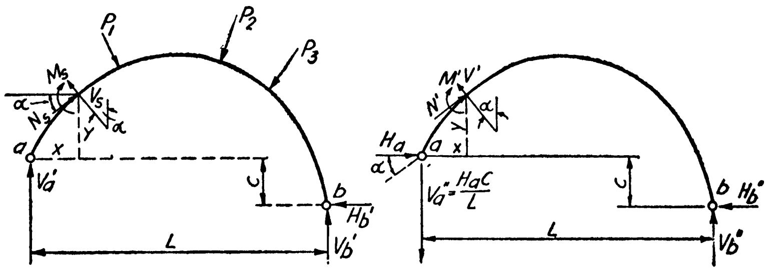


FIG. 125b and c.

which, by means of equation 107, can be expressed in the form

$$\Delta_a = \int_a^b \frac{(M_s + M')\dfrac{\partial M'}{\partial H_a}\,ds}{EI} + \int_a^b \frac{(N_s + N')\dfrac{\partial N'}{\partial H_a}\,ds}{AE} \tag{108a}$$

But, since

$$M' = -H_a\left(y + \frac{c}{L}x\right)$$

and

$$N' = H_a \cos\alpha - H_a \frac{c}{L}\sin\alpha$$

therefore

$$\frac{\partial M'}{\partial H_a} = -\left(y + \frac{c}{L}x\right)$$

$$\frac{\partial N'}{\partial H_a} = \cos\alpha - \frac{c}{L}\sin\alpha$$

If the values of $\dfrac{\partial M'}{\partial H_a}$ and $\dfrac{\partial N'}{\partial H_a}$ are substituted in equation 108a and the summation is used for the integration, the expression for Δ_a becomes

$$\Delta_a = \frac{1}{E}\left\{\sum_a^b - M_s\left(y + \frac{c}{L}x\right)\frac{\Delta s}{I} + \sum_a^b N_s\left(\cos\alpha - \frac{c}{L}\sin\alpha\right)\frac{\Delta s}{A}\right.$$

$$\left. + H_a\sum_a^b\left(y + \frac{c}{L}x\right)^2\frac{\Delta s}{I} + H_a\sum_a^b\left(\cos\alpha - \frac{c}{L}\sin\alpha\right)^2\frac{\Delta s}{A}\right\} \tag{108b}$$

When Δ_a is equal to zero, the value of H_a is

$$H_a = \frac{\displaystyle\sum_a^b M_s\left(y + \frac{c}{L}x\right)\frac{\Delta s}{I} - \sum_a^b N_s\left(\cos\alpha - \frac{c}{L}\sin\alpha\right)\frac{\Delta s}{A}}{\displaystyle\sum_a^b\left(y + \frac{c}{L}x\right)^2\frac{\Delta s}{I} + \sum_a^b\left(\cos\alpha - \frac{c}{L}\sin\alpha\right)^2\frac{\Delta s}{A}} \tag{109a}$$

If the deformation due to the normal force is neglected and the supports are at the same elevation, that is c is equal to zero, then equation 109a reduces to the simple expression

$$H_a = \frac{\sum_a^b \frac{M_s y \, \Delta s}{I}}{\sum_a^b \frac{y^2 \, \Delta s}{I}} \tag{109b}$$

Both numerator and denominator of equation 109b can be calculated for any applied loading on a given arch.

For arches with low $\frac{\text{rise}}{\text{span}}$ ratios, that is, flat arches, the value of H is obtained more accurately by retaining the expression for the normal force in the denominator but omitting it in the numerator. If this is done, the calculated value of H, which is always less than when only the bending moment is considered, is practically equal to the true value. For this assumption, the expression for H_a becomes

$$H_a = \frac{\sum_a^b \frac{M_s y \, \Delta s}{I}}{\sum_a^b \frac{y^2 \Delta s}{I} + \sum_a^b \cos^2 \alpha \frac{\Delta s}{A}} \tag{109c}$$

71. Influence Diagram for H_a. The value of H_a for a unit vertical load anywhere on a two-hinged arch can be calculated by equations

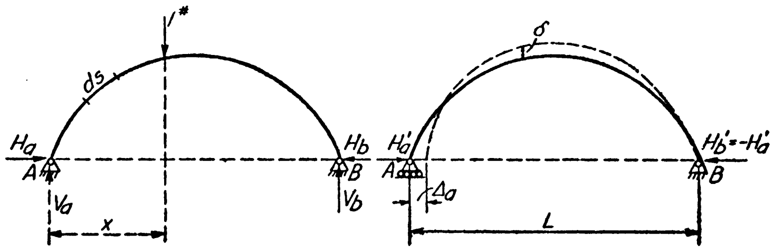


FIG. 126a and b.

109a, b, and c. In these equations, the denominator is a constant as it is independent of the position of the load. However, as the numerator must be calculated for each position of the unit load, this procedure is likely to be laborious for most arches, particularly if the arch is unsymmetrical and has variable cross section. For such problems a graph-

ical solution that is based on the reciprocal theorem is more convenient. This solution follows the general form for this type of problem, that is an auxiliary force system consisting of two unit horizontal forces applied at the points A and B, Fig. 126b, is placed on the arch. This force system causes the displacements shown by the dotted line, for which, according to the reciprocal theorem,

$$H_a \Delta_a - P\delta = 0$$

or

$$H_a = P \frac{\delta}{\Delta_a} \tag{110}$$

The ordinates to the influence diagram are therefore equal to $\frac{\delta}{\Delta_a}$, which can be obtained in the following manner. Consider the $\frac{y\,\Delta s}{I}$ value for each element of the arch as an applied weight F on a beam $A'B'$ whose span is equal to L (Fig. 127a). After calculating the reac-

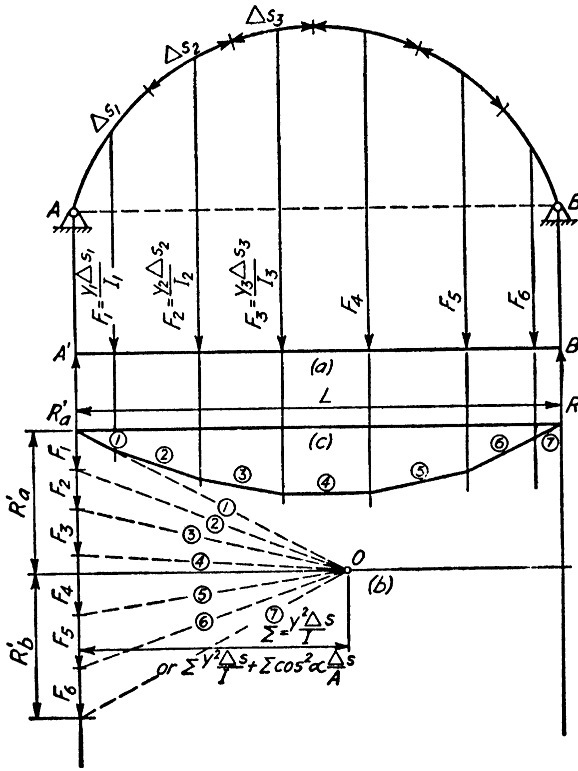


FIG. 127.

tions R'_a and R'_b for this beam, draw a force polygon, Fig. 127*b*, whose pole O is at a distance

$$\Delta_a = \sum_A^B \frac{y^2 \Delta s}{I} + \sum_A^B \frac{\cos^2 \alpha \Delta s}{A} \tag{111}$$

from the vertical forces F_1, F_2, $F_3 \cdots$. The funicular polygon Fig. 127*c* is the influence diagram for H_a to the same scale as the span L of the beam $A'B'$ is drawn.

The proof of the above construction can be verified by considering the displacements caused by the rotation of any element Δs, as shown in

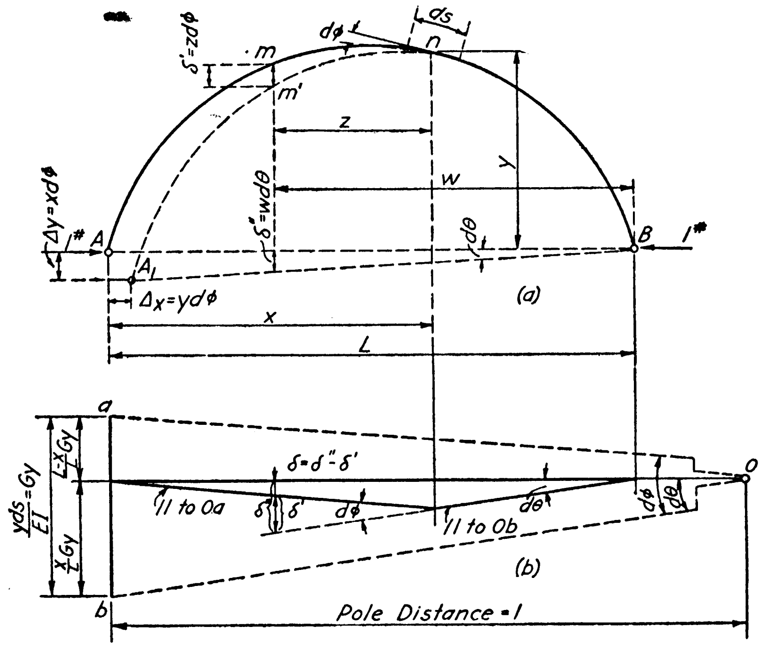


FIG. 128.

Fig. 128*a*. Thus the vertical displacement of point m relative to point n due to the rotation $d\phi$ of the element Δs is mm', or

$$\delta' = z\, d\phi = z \frac{M \Delta s}{EI} = z \frac{1 \text{ lb } (y \Delta s)}{EI}$$

However, the actual vertical displacement with respect to the fixed point B is equal to δ' minus the displacement that is produced when the entire arch is rotated about B through an angle $d\theta$ which will bring

point A back on the line AB. That is, the true vertical displacement δ of point m is equal to δ' minus δ'' or

$$\delta = \delta' - \delta'' = z\,d\phi - w\,d\theta \tag{112}$$

The angle $d\phi$ can be constructed graphically by the tangent method, Fig. 128*b*, in which the ordinate is equal to $\dfrac{y\,ds}{EI}$ and the abscissa is unity. The angle $d\theta$ has $\dfrac{\Delta_y}{L}$ or $\dfrac{x\left(\dfrac{y\,ds}{EI}\right)}{L}$ as ordinate and unity as a base. Therefore, if these angles are drawn as shown in Fig. 128*b*, the difference of δ' and δ'', or the vertical displacement δ, is obtained graphically. However, the ordinate to the influence diagram is $\dfrac{\delta}{\Delta_{ax}}$, and, therefore, instead of unity the base of the angles must be equal to Δ_{ax} or $\dfrac{y^2\,ds}{EI}$. As E is a constant it can be omitted from all the terms. This construction can be extended so as to prove the general graphical solution described above for any number of elements Δs.

Example 42. The graphical construction of an influence diagram for the horizontal reaction H for a two-hinged elliptical arch is illustrated in Fig. 129. As the arch has a constant cross section the term $\dfrac{\Delta s}{I}$ is a constant for equal length elements and therefore the vertical ordinate y at the center of the element can be used for the elastic weight $\dfrac{y\,\Delta s}{I}$. These elastic weights are shown as applied forces on the arch axis in Fig. 129*a* and are used to draw the force polygon in Fig. 129*b*. The pole distance has been taken equal to

$$\frac{\Delta_{ax}}{10} = \frac{\sum \dfrac{y^2\,\Delta s}{I}}{10} = \frac{\Sigma y^2}{10} = 143.5$$

The term $\sum \dfrac{\cos^2 \alpha\,\Delta s}{A}$, which represents the effect of the normal force, has been neglected.

The equilibrium polygon shown by the solid line in Fig. 129*c* that was drawn from the force polygon in Fig. 129*b* is the influence diagram for H. The scale of this diagram is 10 times the scale of the arch. If the arch is drawn to 1 in. equals 10 ft the scale for H is 1 in. equals 1 lb.

The broken line in Fig. 129c gives the values of H for a parabolic arch with the same $\frac{\text{rise}}{\text{span}}$ ratio.

The influence diagram for the bending moment at any section can be obtained by superimposing the moment due to H upon the moment

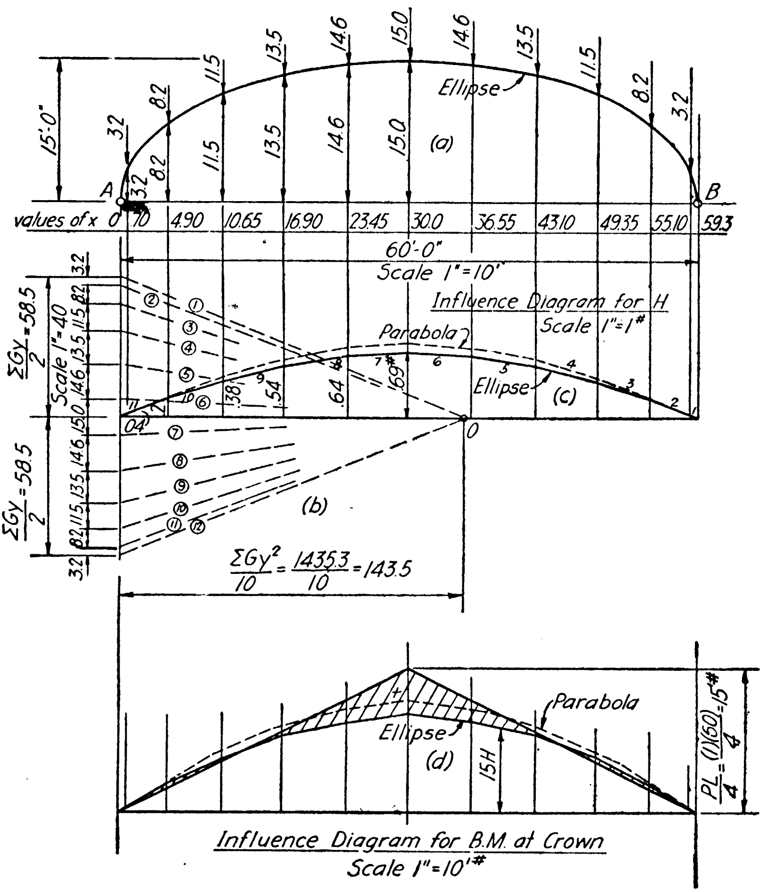


FIG. 129.

for the simply supported structure. The bending moment at the crown is shown in Fig. 129d, for both the elliptical and parabolic arches. The ordinates of these diagrams are equal to

$$M_{\text{crown}} = M_s - 15H$$

72. Fixed-End Arches. The analysis of any arch that is restrained at both supports, Fig. 130, is usually started by resolving the external

forces into known and unknown force systems, the unknown forces, of course, being in terms of the redundant quantities M_a, V_a, and H_a.

The bending moment at any element ds is

$$M = M_a + V_a x - H_a y - P_1 d_1 - P_2 d_2 \tag{113}$$

and the total strain energy due to the bending moments is

$$U = \int_a^b \frac{M^2\, ds}{2EI} = \int_a^b \frac{(M_a + V_a x - H_a y - M')^2\, ds}{2EI} \tag{114}$$

where $M' = P_1 d_1 + P_2 d_2$ = moment of applied loads to the left of the element ds.

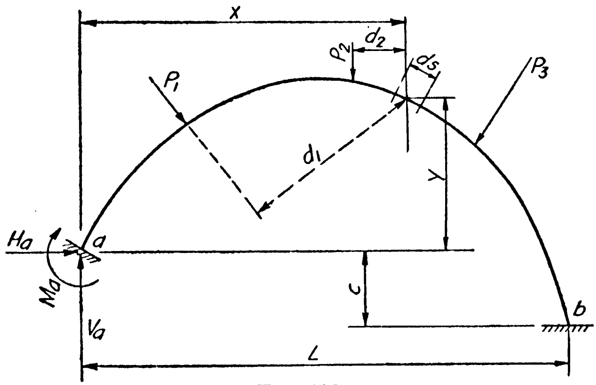


FIG. 130.

The linear and angular displacements of point a according to Castigliano's theorem are

$$\Delta_{ax} = \frac{\partial U}{\partial H_a} = \int_a^b \frac{(M_a + V_a x - H_a y - M')(-y)\, ds}{EI} \tag{115a}$$

$$\Delta_{ay} = \frac{\partial U}{\partial V_a} = \int_a^b \frac{(M_a + V_a x - H_a y - M')(x)\, ds}{EI} \tag{115b}$$

$$\theta_a = \frac{\partial U}{\partial M_a} = \int_a^b \frac{(M_a + V_a x - H_a y - M')\, ds}{EI} \tag{115c}$$

By substituting G for $\frac{\Delta s}{I}$ and replacing the integration by summation, the above equations can be written in the convenient form

$$E\Delta_{ax} = -M_a \Sigma G y - V_a \Sigma G x y + H_a \Sigma G y^2 + \Sigma M' G y \tag{116a}$$

$$E\Delta_{ay} = M_a \Sigma G x + V_a \Sigma G x^2 - H_a \Sigma G x y - \Sigma M' G x \tag{116b}$$

$$E\theta_a = M_a \Sigma G + V_a \Sigma G x - H_a \Sigma G y - \Sigma M' G \tag{116c}$$

If the arch is fixed at a, then $\Delta_{ax} = \Delta_{ay} = \theta_a = 0$, and equations 116$a$, b, and c can be solved for the redundant forces H_a, V_a, and M_a. This operation is laborious but involves no particular difficulty.

Example 43. To illustrate the use of equations 116a, b, and c let us assume the elliptical arch that was used in Example 42 to be fixed at

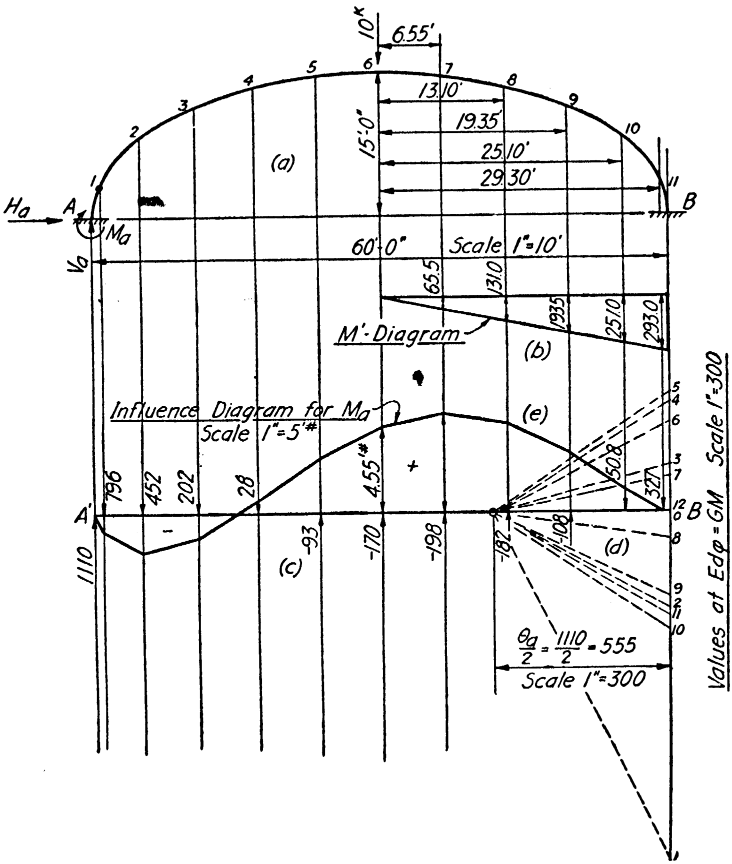


FIG. 131.

both ends, Fig. 131a. In this arch the values of $\frac{\Delta s}{I} = G$ are constant for each element and can be taken equal to unity. The value of H_a, V_a, M_a will be calculated for a concentrated vertical load of 10 kips at the center of the span. The values of the various summations are

$$\Sigma x = 330 \qquad \Sigma y = 117.0 \qquad \Sigma xy = 3511$$

$$\Sigma x^2 = 14{,}059 \qquad \Sigma y^2 = 1435.3$$

The value of the terms $\Sigma M'G$, $\Sigma M'Gy$, and $\Sigma M'Gx$ depend upon the applied load, which, in this problem, is a concentrated vertical load at the center of the span.

The variation of M' is shown in Fig. 131*b*, and the values of the summations are

$$\Sigma M'G = (65.5 + 131.0 + 193.5 + 251.0 + 293.0) = 934$$

Values of x = 36.55, 43.10, 49.35, 55.10, and 59.30.

$$\Sigma M'Gx = 2394 + 5646 + 9549 + 13{,}830 + 17{,}375 = 48{,}794$$

Values of y = 14.6, 13.5, 11.5, 8.2, and 3.2.

$$\Sigma M'Gy = 956 + 1769 + 2225 + 2058 + 938 = 7946$$

When the above values are substituted in equations 116*a*, *b*, and *c* we obtain

$$E\Delta_{ax} = -117M_a - 3511V_a + 1435H_a + 7946 = 0$$

$$E\Delta_{ay} = 330M_a + 14{,}059V_a - 3511H_a - 48{,}794 = 0$$

$$E\theta_a = 11M_a + 330V_a - 117H_a - 934 = 0$$

Solving, these equations give

$$H_a = 10.37 \text{ kips} \qquad V_a = 5.0 \text{ kips} \qquad M_a = 45.2 \text{ ft-kips}$$

73. Influence Diagram for M_a. If the force system shown in Fig. 132 is applied to the arch, the elastic curve can be used to obtain the influence diagram for M_a. By the reciprocal theorem

$$M_a\theta_a - P\delta = 0$$

or

$$M_a = P\frac{\delta}{\theta_a}$$

From equations 116*a* and *b*

$$E\Delta_{ax} = -M'_a\Sigma Gy - V'_a\Sigma Gxy + H'_a\Sigma Gy^2 = 0 \qquad (117a)$$

$$E\Delta_{ay} = M'_a\Sigma Gx + V'_a\Sigma Gx^2 - H'_a\Sigma Gxy = 0 \qquad (117b)$$

Equations 117*a* and *b* will give the value of V'_a and H'_a for any value of M'_a. The value of $E\theta_a$ can then be obtained from equation 116*c*, or

$$E\theta_a = M'_a\Sigma G + V'_a\Sigma Gx - H'_a\Sigma Gy \qquad (117c)$$

For any value of M_a' the bending moment M on an element ds, Fig. 132, is

$$M = M_a' + V_a'x - H_a'y$$

and the rotation $d\phi$ of each element is

$$d\phi = \frac{M\,ds}{EI} \qquad \text{or} \qquad E\,d\phi = GM$$

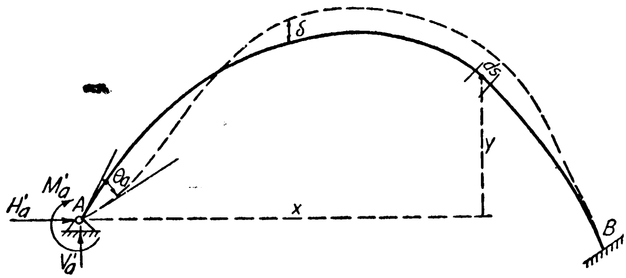


FIG. 132.

If each value of $E\,d\phi$ or GM is used as an applied force and $E\theta_a$ is used as a pole distance, the funicular polygon will be the influence diagram for M_a. The proof of this construction is practically the same as that in Article 71 for the two-hinged arch. However, here the tangent at B is fixed and therefore no correction for rotation is necessary.

Example 44. An influence diagram for the end moment of the elliptical arch used in Examples 42 and 43 will be constructed by the semi-graphical method.

Substituting the values of the summations that are given in Example 43 into equations 117*a* and *b* gives

$$-117M_a' - 3511V_a' + 1435H_a' = 0$$

$$330M_a' + 14{,}059V_a' - 3511H_a' = 0$$

from which

$$V_a' = -0.008M_a' \qquad H_a' = 0.062M_a'$$

and from equation 117*c*

$$E\theta_a = 11M_a' + (-0.008M_a')(330) - (0.062M_a')(117)$$

or

$$E\theta_a = 11M_a' - 2.64M_a' - 7.25M_a' = 1.11M_a'$$

If the value of M'_a is taken equal to 1000, then $V'_a = -8$, $H'_a = 62$, and $E\theta_a = 1110$. The value of M at each element is given in Table 9.

TABLE 9

Point	x	y	$-8x$	$-62y$	$M = 1000 - 8x - 62y$
1	0.70	3.2	−5.6	−198.4	796
2	4.9	8.2	−39.2	−508.4	452.4
3	10.65	11.5	−85.2	−713.0	201.8
4	16.90	13.5	−135.2	−837.0	27.8
5	23.45	14.6	−187.6	−905.2	−92.8
6	30.0	15.0	−240.0	−930.0	−170.0
7	36.55	14.6	−292.4	−905.2	−197.6
8	43.10	13.5	−344.8	−837.0	−181.8
9	49.35	11.5	−394.8	−713.0	−107.8
10	55.10	8.2	−440.8	−508.4	50.8
11	59.30	3.2	−474.4	−198.4	327.2

The value of $E\,d\phi = GM = M$ is used as an elastic weight on the beam $A'B'$, Fig. 131*c*, and a force polygon, Fig. 131*d*, is constructed for these forces. Since $\theta_b = 0$, the reaction of the conjugate beam is zero at B'. A pole distance equal to

$$\frac{E\theta_a}{2} = \frac{1110}{2} = 555$$

has been adopted for the construction of the funicular polygon in Fig. 131*e*, which is the influence diagram for M_a. The ordinates to this diagram are to twice the scale of the span of the beam $A'B'$. If $\dfrac{E\theta_a}{n}$ is taken as the pole distance, the scale of the influence diagram is n times the scale of the span.

74. Influence Diagram for H_a. An influence diagram for the horizontal reaction of a fixed-end arch can also be obtained by a graphical solution. If the auxiliary force system shown in Fig. 133*a* is applied to the arch so as to give point A a horizontal displacement Δ_{ax} but with θ_a and Δ_{ay} equal to zero, then, by equations 118*a* and 118*b*, the reactions V'_a and M'_a can be calculated for any value of H'_a.

$$E\Delta_{ay} = M'_a\Sigma Gx + V'_a\Sigma Gx^2 - H'_a\Sigma Gxy = 0 \qquad (118a)$$

$$E\theta_a = M'_a\Sigma G + V'_a\Sigma Gx - H'_a\Sigma Gy = 0 \qquad (118b)$$

The horizontal displacement $E\Delta_{ax}$ can now be calculated for any value of H'_a by equation 118*c*.

$$E\Delta_{ax} = -M'_a\Sigma Gy - V'_a\Sigma Gxy + H'_a\Sigma Gy^2 \qquad (118c)$$

The bending moment at any element of the arch in Fig. 133*a* is now known since

$$M = M'_a - H'_a y + V'_a x \tag{119}$$

The vertical reaction V'_a will, of course, be zero for a symmetrical arch; otherwise the influence diagram for H_a would not be symmetrical.

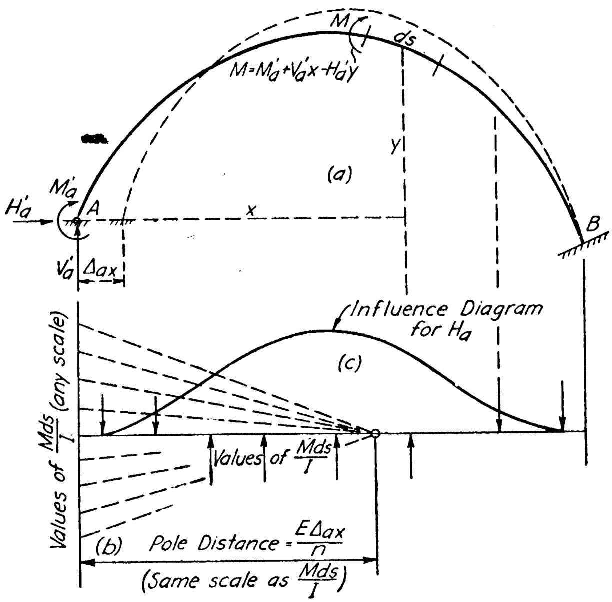


FIG. 133.

The influence diagram for H_a is now drawn by laying off a force polygon, Fig. 133*b*, in which the $\dfrac{M\,ds}{I}$ values are the elastic weights and the pole distance is equal to $\dfrac{E\,\Delta_{ax}}{n}$ (n is any convenient number). These quantities are drawn to any scale.

The funicular polygon, Fig. 133*c*, that is drawn from the force polygon, Fig. 133*b*, gives the influence diagram for H_a to n times the scale to which the span of the arch was drawn.

Example 45. An influence diagram for the horizontal reaction H_a of the elliptical arch used in Examples 42, 43, and 44 will be constructed by the method that has just been explained.

The values of the various summations for this arch are given in

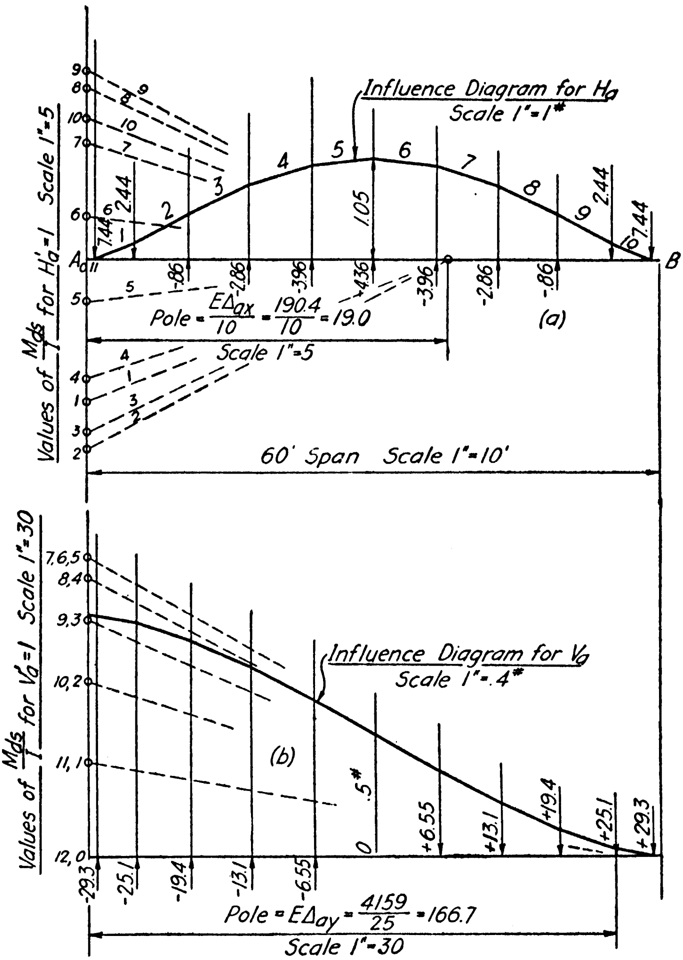


FIG. 134.

Example 43. If these values are substituted in equations 118*a* and *b* we obtain

$$E\Delta_{ay} = 330M_a' + 14{,}059V_a' - 3511H_a' = 0$$

$$E\theta_a = 11M_a' + 330V_a' - 117H_a' = 0$$

from which

$$M_a' = 10.64H_a' \qquad \text{and} \qquad V_a' = 0$$

When the above values of M_a' and V_a' are substituted in equation 118*c* the expression for $E\Delta_{ax}$ becomes

$$E\Delta_{ax} = -(10.64H_a')(117) + H_a'(1435.3) = 190.4H_a'$$

The bending moment at any element of the arch for H'_a equal to unity is

$$M = M'_a - H'_a y = 10.64H'_a - H'_a y = 10.64 - y$$

The value of G or $\frac{\Delta s}{I}$ is taken as unity for each element. The end points A and B must not be considered, as they are not the mid-points of an element. In Fig. 134a, the values of $\frac{M\ \Delta s}{I}$ are represented as forces or elastic weights from which the force polygon is drawn with a scale of 1 in. equal to 5 units. The pole distance is made equal to

$$\frac{E\Delta_{ax}}{10} = \frac{190.4}{10} = 19.04$$

and is drawn to the same scale. The funicular polygon constructed from the force polygon is the influence diagram for H_a. The scale of the arch span is 1 in. equals 10 ft, and therefore the scale of the influence diagram is 1 in. $= \frac{10}{n} = \frac{10}{10} = 1$ lb. The ordinate at the center of the span scales about 1.05 lb, which practically checks the result of the algebraic solution of Example 43.

75. Influence Diagram for V_a. An influence diagram for V_a can be drawn in the same manner as just described for M_a and H_a if the force system shown in Fig. 135a is applied to the arch. This force system

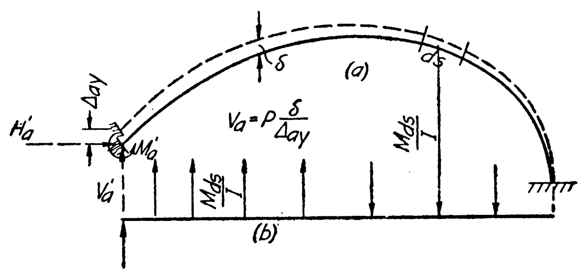


FIG. 135.

must give some vertical displacement Δ_{ay} at point A while Δ_{ax} and θ_a are kept equal to zero. The forces H'_a and M'_a that are necessary to give these displacements can be calculated from equations 120a and b in terms of V'_a:

$$E\Delta_{ax} = -M'_a\Sigma Gy - V'_a\Sigma Gxy + H'_a\Sigma Gy^2 = 0 \qquad (120a)$$

$$E\theta_a = M'_a\Sigma G + V'_a\Sigma Gx - H'_a\Sigma GY = 0 \qquad (120b)$$

The vertical displacement $E\Delta_{ay}$ can then be calculated from equation 120c.

$$E\Delta_{ay} = M'_a\Sigma Gx + V'_a\Sigma Gx^2 - H'_a\Sigma Gxy \qquad (120c)$$

The bending moment at any section, as for the preceding problems, is equal to

$$M = M'_a - H'_a y + V'_a x$$

For a symmetrical arch H'_a is equal to zero and

$$M'_a = -\frac{V'_a L}{2} \qquad (120d)$$

where L is the span of the arch. Therefore, for a symmetrical arch

$$M = V'_a\left(x - \frac{L}{2}\right) \qquad (120e)$$

Again the values of $\frac{M\ \Delta s}{I}$, Fig. 135*b*, are used to lay off a force polygon from which the influence diagram for V_a is constructed. The pole distance must be equal to $\frac{E\Delta_{ay}}{n}$, in which n is any convenient number. The force polygon can be constructed to any scale. The influence diagram for V_a will, of course, be n times the scale to which the arch span is drawn.

Example 46. If the summations for the elliptical arch that has been used in Examples 43, 44, and 45 are substituted in equations 120*a* and *b* the expressions for $E\Delta_{ax}$ and $E\theta_a$ become

$$E\Delta_{ax} = -117M'_a - 3511V'_a + 1435.3H'_a = 0$$

$$E\theta_a = 11M'_a + 330V'_a - 117H'_a = 0$$

from which $H'_a = 0$, $M'_a = -30V'_a = -\frac{V'_a L}{2}$.

Therefore, if V'_a is taken equal to unity, $M'_a = -30$ and the bending moment M at any element is

$$M = (x - 30)$$

The values of $\frac{M\ \Delta s}{I}$ or M are represented as forces in Fig. 134*b*, and these values are used to draw the force polygon. The pole distance is taken equal to $\frac{E\Delta_{ay}}{25}$, which is found to be 166.7 by equation 120c.

The influence diagram for V_a which is drawn from the rays of the force polygon is therefore to a scale of 1 in. $= \frac{10}{25} = 0.4$ lb. The ordinate at the center must be equal to 0.5 lb and at A should be 1 lb.

76. Effect of Temperature Change. Any change in temperature produces a volumetric change which in an arch may cause stresses of considerable magnitude if the supports resist this motion. Thus, in any arch, Fig. 136, which is fixed at B and free to move at A, a rise in temperature will cause the end section mn to move to some position $m'n'$.

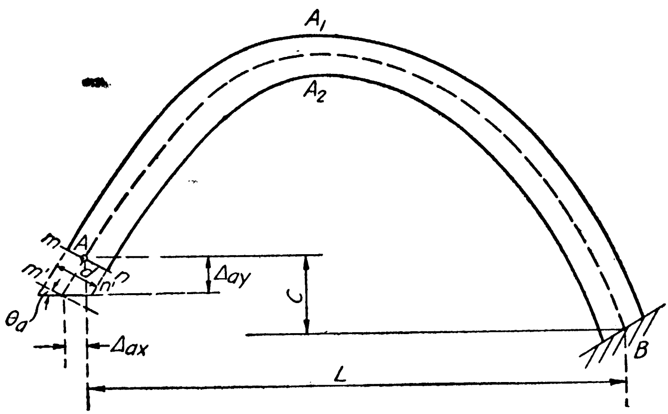


FIG. 136.

This motion is expressed most conveniently in terms of the horizontal displacement Δ_{ax}, the vertical displacement Δ_{ay}, and the rotation θ_a. These displacements are caused by the change in length of the horizontal projection of the arch axis, the vertical projection of the axis, and the relative change in length of the intradosal and extradosal surfaces, respectively. These movements are easily calculated by the expressions

$$\Delta_{ax} = \alpha t L \tag{121a}$$

$$\Delta_{ay} = \alpha t c \tag{121b}$$

$$\theta_a = \frac{\alpha t(mm' - nn')}{d} \tag{121c}$$

where α = coefficient of linear expansion.
t = change in temperature.
mm' = movement of point m.
nn' = movement of n.
d = depth of arch at point A.
c = difference in elevation of supports A and B.
L = horizontal projection of axis.

In most arches, the end rotation θ_a can be taken equal to zero as it depends primarily on vertical motion, which is usually small. Also, the vertical displacement Δ_{ay} is zero for symmetrical arches. The values of H_a, M_a, and V_a necessary to prevent motion of the cross section at A can be calculated directly from equations 116a, b, and c. When the supports are at the same elevation, that is with Δ_{ay} and θ_a equal to zero, the value of H_a is given directly by the solution of equations 118a, b, and c. These equations have already been solved in the construction of the influence diagram for H_a.

Example 47. The reactions and moment at the crown of the elliptical arch used in the preceding examples will be calculated for a rise in temperature of 50° F. The following data are used in the calculations:

$$t = 50^\circ \qquad \alpha = 6 \times 10^{-6} \qquad E = 2 \times 10^6 \text{ lb per in.}^2$$

$$I = 0.667 \text{ ft}^4 \qquad G = \frac{\Delta s}{I} = \frac{6.8}{0.667} = 10.2$$

$$E\Delta_{ax} = E\alpha tL = 5.17 \times 10^6 \text{ lb per ft.}$$

From Example 45, the value of $E\Delta_{ax}$ is equal to

$$E\Delta_{ax} = 190.4H_aG$$

when $\Delta_{ay} = \theta_a = 0$. Therefore

$$(190.4H_a)(10.2) = (5.17)(10^6)$$

$$H_a = 2660 \text{ lb}$$

$$M_a = 10.64H_a = 28{,}300 \text{ ft-lb}$$

$$M \text{ (at crown)} = 28{,}300 - (2660)(15) = -11{,}700 \text{ ft-lb}$$

For a temperature drop, the signs of the forces would be reversed.

77. Relation between End Forces and End Displacements. In the preceding articles, equations have been developed for determining the relation between end forces and a given end displacement. Thus, by means of equations 117a, b, and c, the end forces M'_a, H'_a, and V'_a can be expressed in terms of the end rotation $E\theta_a$, when $E\Delta_{ax}$ and $E\Delta_{ay}$ are equal to zero. In Example 44, these equations have the numerical values

$$E\theta_a = 1.11M'_aG \qquad \text{or} \qquad M'_a = \frac{0.9E\theta_a}{G}$$

$$H'_a = 0.062M'_a = \frac{0.0558E\theta_a}{G} \qquad V'_a = -0.008M'_a$$

By means of equation 118*a*, *b*, and *c*, the end forces M'_a, H'_a, and V'_a are expressed in terms of a given horizontal displacement $E\Delta_{ax}$ when $E\Delta_{ay}$ and $E\theta_a$ are set equal to zero. In example 45, these equations gave numerical values of

$$E\Delta_{ax} = 190.4H'_aG \qquad \text{or} \qquad H'_a = \frac{E\Delta_{ax}}{190.4G}$$

and

$$M'_a = 10.64H'_a = 0.0558\frac{E\Delta_{ax}}{G}$$

From these relations the moment and horizontal component at the support can be calculated for a given horizontal displacement.

78. Stiffness and Carry-Over Factors for Curved Members. The terms stiffness and carry-over factors have the same meaning for a curved beam as for a straight one. That is, if the end moment M'_a is equal to

$$M'_a = C_aE\theta_a \tag{122a}$$

C_a is the stiffness factor with respect to a rotation at A. The moment at B can be expressed in terms of M'_a or θ_a, that is,

$$M'_{ba} = C_{ab}M'_{ab} \tag{122b}$$

where C_{ab} is the carry-over factor. For example, we have already seen that for the elliptical arch used in preceding problems

$$M'_a = \frac{0.9E\theta_a}{G} \qquad \text{or} \qquad C_a = \frac{0.9}{G}$$

and $M'_b = -M'_a + V'_aL = -M'_a + (0.008M'_a)(60) = -0.52M'_a$. Therefore the stiffness factor is $\frac{0.9}{G}$, and the carry-over factor is -0.52.

79. Distribution Factors. After the stiffness coefficients have been calculated the distribution factors are easily determined for any number of members, either straight or curved, that meet at a joint. For example, if the three members shown in Fig. 137 are considered continuous at joint b, the end moments for all members can be expressed in terms of the fixed-end moments and the moments due to the rotation θ_b and horizontal displacement Δ_{xb}.

The difference between a frame composed of straight members and one with curved beams consists primarily in the larger horizontal force applied to the joint from the curved members and the possibility of joint b having a relative horizontal displacement with respect to a and c. For this reason there will usually be more unknown displacements to

consider for frames with curved members than for frames with straight ones. It is always necessary to assume first that an auxiliary force or reaction is applied at the joint so as to prevent any horizontal movement. In other words, if the moment-distribution method is to be used, the joint must not be permitted to translate while it is given a rotation.

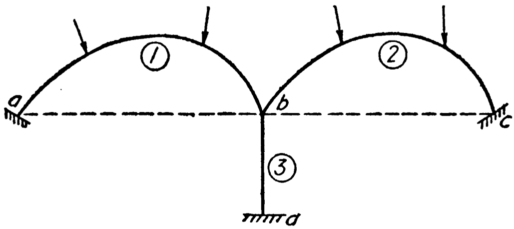


FIG. 137.

If the members in Fig. 137 undergo only a rotation at joint b,

$$M_{ba} = M_{Fba} + C_{b1}E\theta_b = M_{Fba} + M'_{ba}$$

$$M_{bc} = M_{Fbc} + C_{b2}E\theta_b = M_{Fbc} + M'_{bc}$$

$$M_{bd} = C_{b3}E\theta_b = M'_{bd}$$

To satisfy the equilibrium condition at joint b,

$$M_{ba} + M_{bc} + M_{bd} = 0$$

or

$$M_{Fba} + C_{b1}E\theta_b + M_{Fbc} + C_{b2}E\theta_b + C_{b3}E\theta_b = 0$$

from which

$$E\theta_b = -\frac{M_{Fba} + M_{Fbc}}{C_{b1} + C_{b2} + C_{b3}} = -\frac{\Sigma M_{Fb}}{\Sigma C_b} \tag{123}$$

Therefore, the corrections to the fixed-end moments are

$$M'_{ba} = \frac{C_{b1}}{\Sigma C_b}(-\Sigma M_{Fb}) \tag{124a}$$

$$M'_{bc} = \frac{C_{b2}}{\Sigma C_b}(-\Sigma M_{Fb}) \tag{124b}$$

$$M'_{bd} = \frac{C_{b3}}{\Sigma C_b}(-\Sigma M_{Fb}) \tag{124c}$$

Example 48. The reactions for the frame in Fig. 138a will be calculated for a uniform load of 1 kip per foot by means of the moment-distribution method. This method of analysis has no particular merit over

other procedures for one-span frames or for one type of loading. When the problem involves several spans and various loading arrangements, however, the moment-distribution method, if thoroughly understood, is often the most advantageous.

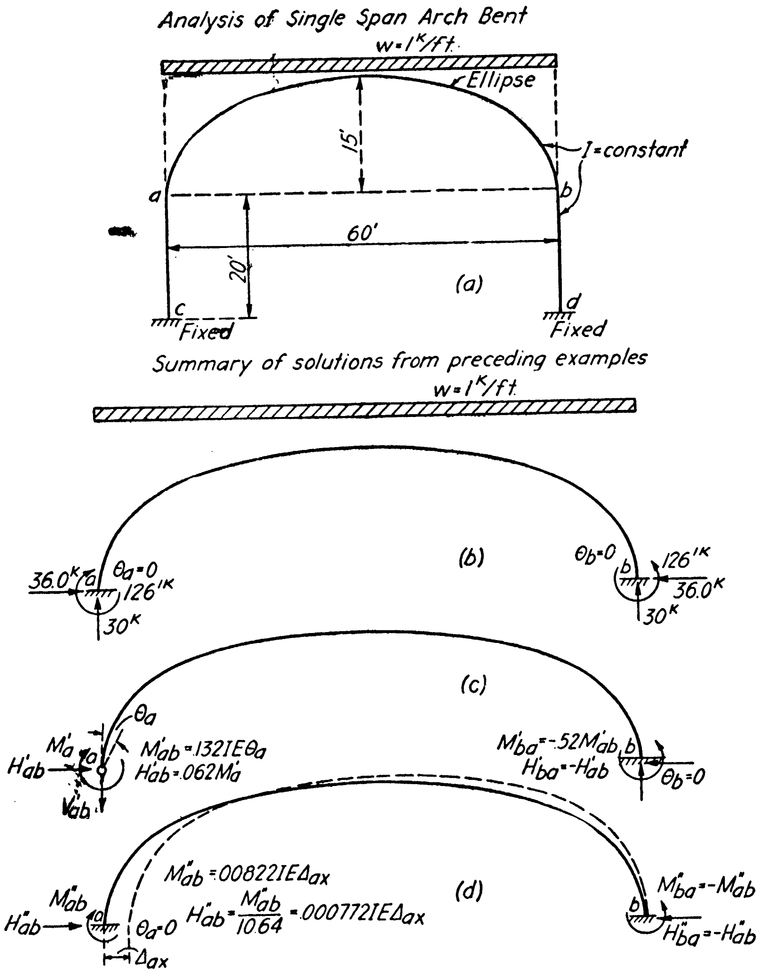


FIG. 138a–d.

The solution of the problem must begin with the determination of the reactions for the arch member for full restraint at points a and b. If the graphical constructions that have been explained and illustrated are used for this part of the problem, the necessary data for calculating the stiffness and carry-over factors will also have been obtained. This

sequence of numerical operations is illustrated in the present problem as the member *ab* is identical with the elliptical arch for which the influence diagrams for M_a, H_a, and V_a were drawn in Examples 44, 45, and 46, respectively. From Example 44 we obtain not only the fixed-end moments for any type of vertical loads but also the stiffness and carry-over factors. The magnitude and direction of the fixed-end moments which are determined from the area of the influence diagram for M_a in Fig. 131*e* are

$$M_{Fab} = +126 \text{ ft-kips}$$

$$M_{Fba} = -126 \text{ ft-kips}$$

It should be noted that these signs are the reverse of those for a straight beam. In Example 44 the rotation θ_a for any moment M'_a applied at end *a* with end *b* fully restrained was found to be

$$E\theta_a = 1.11M'_a$$

However, in that solution the relative value of $\frac{\Delta s}{I}$ was taken equal to unity whereas the actual value is approximately $\frac{6.8}{I}$ (scaled). Therefore,

$$E\theta_a = 1.11\left(\frac{6.8}{I}\right)M'_a$$

or

$$M'_a = 0.132EI\theta_a$$

giving the stiffness factor as $0.132I$.

In the same solution, the corresponding value of H'_a and V'_a were found to be

$$H'_a = 0.062M'_a$$

$$V'_a = -0.008M'_a$$

The value of the fixed-end moment at *b* for a moment applied at *a* is therefore (taking summation of moments about *b* equal to zero)

$$M'_a + M'_b - (0.008M'_a)(60) = 0$$

or

$$M'_b = -M'_a + 0.48M'_a = -0.52M'_a$$

Consequently the carry-over factor is -0.52.

From Example 45, the value of the horizontal component H_a is obtained for any vertical load on the arch, and in addition the reactions are expressed in terms of any horizontal motion Δ_{ax}.

The horizontal component H_a for a uniform load of 1 kip per foot is found to be 36.0 kips from the area of the influence diagram that was obtained in Example 45. The relation between the end forces and the horizontal displacement Δ_{ax} was found to be

$$E\Delta_{ax} = 190.4H_a \frac{\Delta s}{I}$$

from which (Fig. 138*d*)

$$H_a'' = \frac{EI\Delta_{ax}}{(190.4)(6.8)} = 0.000772E\Delta_{ax}I$$

The corresponding values of M_a'' and V_a'' are

$$M_a'' = 10.64H_a'' = 0.00822E\Delta_{ax}I$$

$$V_a'' = 0$$

The values of V_a and the relation between the end forces and vertical displacement Δ_{ay} are given in Example 46. As no vertical displacements are considered in the present problem these relations are not needed.

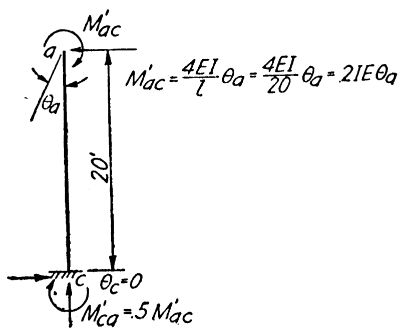


FIG. 138*e*.

The distribution factors at joints *a* and *b* for the arch beam and columns are (see Fig. 138*e*)

MEMBER	CK	r	CARRY-OVER
ac	$0.2I$	0.6	0.5
ab	$0.132I$	0.4	−0.52
	$\Sigma CK = 0.332I$	1.00	

The distribution of the fixed-end moments, which is recorded in Fig. 139*a*, is similar to the procedure for bents with straight members except that the corrections M'_{ab} and M'_{ba} are recorded in a separate column

from the fixed-end moments. This arrangement is necessary as the change in the horizontal component H_a due to the rotation of the joints must also be calculated. The auxiliary forces F prevent any horizontal displacement of joints a and b.

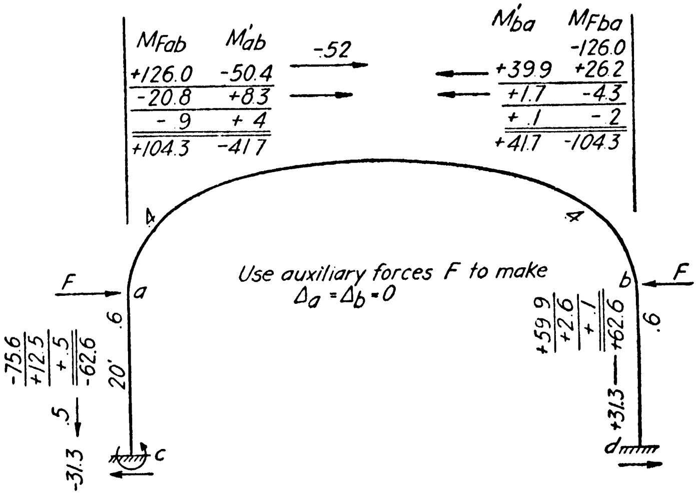


FIG. 139*a*.

From the values recorded in Fig. 139*a* the end moments and shears are equal to

$$M_{ca} = -31.3 \text{ ft-kips} \qquad M_{ac} = -62.6 \text{ ft-kips}$$

$$H_{ca} = \frac{31.3 + 62.6}{20} = 4.7 \text{ kips} \leftarrow$$

$$M_{ab} = 104.3 - 41.7 = 62.6 \text{ ft-kips}$$

$$H_{ab} = \mathrm{H}_{Fab} + 0.062\,(M'_{ab} - M'_{ba})$$

or

$$H_{ab} = 36.0 + 0.062(-41.7 - 41.7) = 30.83 \text{ kips} \rightarrow$$

Therefore, for equilibrium at joint a,

$$F = 30.83 + 4.7 = 35.53 \text{ kips}$$

The two auxiliary forces F must now be removed by the same procedure as was followed in Chapter V for frames with straight members. By this method joints a and b are given any assigned horizontal displacement (say $E\Delta_x = 1000$) while the rotation and vertical displacements are kept zero. As we have already obtained the forces in terms of

the displacement of joint a, these values can be multiplied by 2 to give the values for both displacements, or

$$H_a = -(0.000772EI\Delta_{ax})2 = -1.544I \text{ kips}$$

$$M_{Fab} = -2(0.00822EI\Delta_{ax}) = -16.44I \text{ ft-kips}$$

$$M_{Fba} = +16.44I \text{ ft-kips}$$

$$M_{Fac} = M_{Fca} = \frac{6IE\Delta_{ax}}{(20)(20)} = \frac{6000I}{400} = 15I \text{ ft-kips}$$

The value of I is taken as unity as only the proper relative stiffness is required.

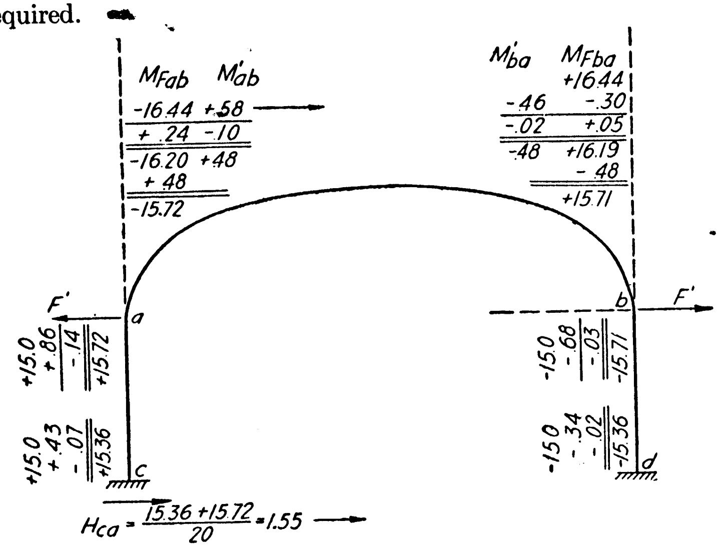


FIG. 139b.

The distribution of these fixed-end moments is recorded in Fig. 139b. The procedure and arrangement are similar to those in Fig. 139a. The value of F' required to give the assigned displacement is therefore

$$F' = H_{ca} + H_{ab}$$

$$H_{ca} = \frac{15.36 + 15.72}{20} = 1.55 \text{ kips} \rightarrow$$

$$H_{ab} = -1.544 + 0.062(0.48 + 0.48) = -1.486 \text{ kips} \leftarrow$$

Therefore

$$F' = 3.036 \text{ kips}$$

To remove the auxiliary forces F in Fig. 139a the moments and shears in Fig. 139b must be multiplied by

$$\frac{35.53}{3.036} = 11.72$$

and added to those in Fig. 139a.

The final values are:

$$M_{ca} = -31.3 + (11.72)(15.36) = 148.7 \text{ ft-kips}$$
$$M_{ab} = 62.6 - (11.72)(15.72) = -121.7 \text{ ft-kips}$$
$$H_c = \frac{148.7 + 121.7}{20} = 13.52 \text{ kips}$$
$$H_{ab} = 30.83 - (11.72)(1.486) = 13.42 \text{ kips}$$
$$M_{\text{crown}} = \frac{wl^2}{8} - 121.7 - (13.42)(15) = 127.0 \text{ ft-kips}$$

***Example 49.* Analysis of a Single-Span Arch Bent for a Concentrated Load at the Center.** The end moments acting on the members of the arch bent that was used in Example 48 will be calculated for a load of 10 kips (Fig. 140) applied at the center of the span.

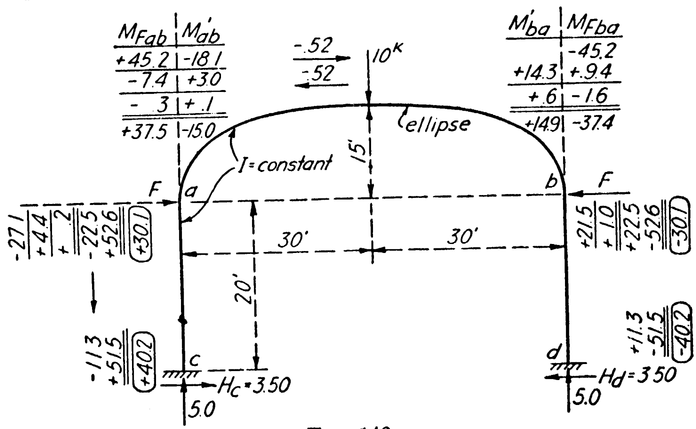


FIG. 140.

The fixed-end forces acting on the arch ab are given by the influence diagrams in Figs. 131e and 134a and b. Therefore, for a 10-kip load at the center of the span,

$$M_{Fab} = -M_{Fba} = (10)(4.52) = 45.2 \text{ ft-kips}$$
$$H_{Fab} = -H_{Fba} = 10.37 \text{ kips}$$
$$V_{Fab} = V_{Fba} = 5.0 \text{ kips}$$

The end forces are first corrected for rotation only, by assuming that the auxiliary forces F, Fig. 140, will prevent any translation of joints a and b. The procedure is similar to that in Example 48.

$$H_{ab} = 10.37 + 0.062(-15.0 - 14.9) = 8.51 \text{ kips}$$

$$H_{ca} = \frac{11.3 + 22.5}{20} = 1.69 \text{ kips}$$

$$F = 8.51 + 1.69 = 10.20 \text{ kips}$$

Multiply moments in Fig. 139b by $\dfrac{10.2}{3.04}$ and add to moments in Fig. 140. The final values are given in the circle. This procedure is similar to that explained in Example 48.

***Example 50.* Analysis of a Continuous Arch Frame.** The continuous arch frame shown in Fig. 141 will be analyzed by successive

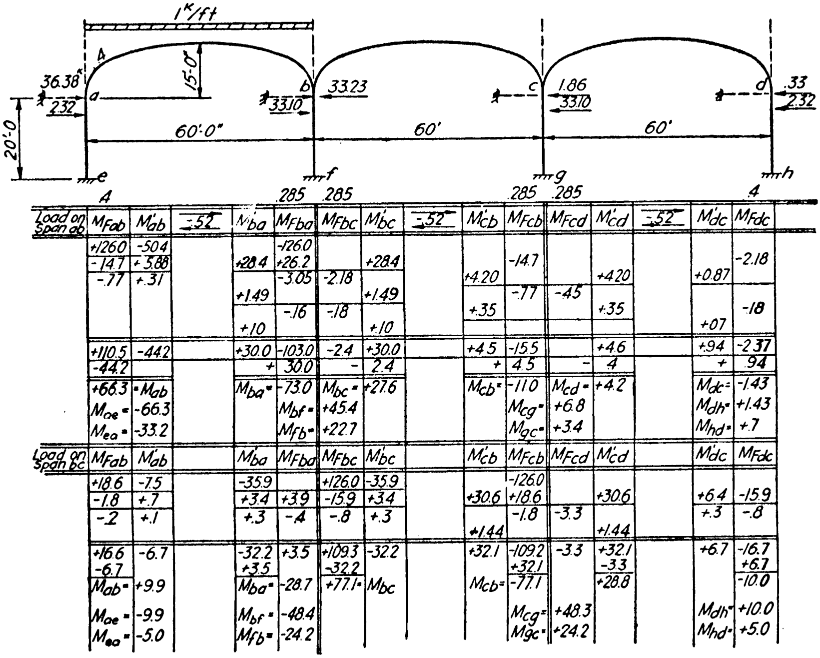


FIG. 141.

approximations for a uniform load first on span ab and then on span bc. The calculations follow the same procedure as in Examples 48 and 49 except that the corrections for the horizontal displacements

of points a, b, c, and d must be made separately. Consequently, five problems must be solved for any general condition of loading, i.e.:

(a) Moments due to applied load with all Δ's = 0 (auxiliary forces applied at a, b, c, and d to prevent translation).

(b) Moments due to any displacement Δ_a.

(c) Moments due to any displacement Δ_b.

(d) Moments due to any displacement Δ_c.

(e) Moments due to any displacement Δ_d.

However, the relationship between the end forces and the Δ's need to be determined only once, as the value of the Δ terms can then be calculated so as to remove any set of auxiliary forces obtained in part (a).

This procedure is best explained by an example. In Fig. 141, the span ab is subjected to a uniform vertical load of 1 kip per horizontal foot. Auxiliary horizontal forces are applied at a, b, c, and d to prevent any translation. If rotations θ_a and θ_b are first assumed equal to zero, then, from previous solutions for a fixed-end arch, we know that the fixed-end moments and horizontal reactions are

$$M_{Fab} = +126.0 \text{ ft-kips} \qquad M_{Fba} = -126.0 \text{ ft-kips}$$

$$H_{Fab} = -H_{Fba} = 36.0 \text{ kips}$$

The distribution factors for joints a and d are given in Example 48. For joints b and c these factors are

Member	C_b	$r = \dfrac{C_b}{\Sigma C_b}$	Carry-Over
ba	$0.132I$	0.285	-0.52
bc	$0.132I$	0.285	-0.52
bf	$0.20I$	0.430	$+0.5$
	$\Sigma C_b = 0.464I$	1.000	

The distribution of the fixed-end moments is carried out in the same manner as in Examples 48 and 49. The corrections due to rotation are kept separate, as these moments change the horizontal components, whereas the moment carried over to the other end is simply a change in the fixed-end moment. After the corrections have become sufficiently small, the final end moments can be obtained in the usual manner. The horizontal forces that act on each member must be determined before the auxiliary forces F_a, F_b, F_c, and F_d can be calculated.

The horizontal forces in the columns can, of course, be calculated directly from the end moments; the horizontal forces acting on the arch beams are equal to (see Example 44)

$$H_{ab} = H_{Fab} + 0.062(M'_{ab} - M'_{ba}) = -H_{ba}$$

That is, the horizontal reaction on the arch is equal to the fixed-end reaction plus 0.062 times the difference of the end moments due to the rotations θ_a and θ_b. Thus, from the values recorded in Fig. 141,

$$H_{ab} = 36.0 + 0.062(-44.2 - 30.0) = 36.0 - 4.6 = 31.4 \rightarrow$$

$$H_e = \frac{66.3 + 33.2}{20} = 4.98 \leftarrow$$

Therefore

$$F_a = 31.4 + 4.98 = 36.38 \text{ kips} \rightarrow$$

$$H_{bc} = 0.062(30.0 - 4.5) = 1.58 \rightarrow$$

$$H_f = \frac{45.4 + 22.7}{20} = 3.41 \rightarrow$$

or

$$F_b = 31.4 + 3.41 - 1.58 = 33.23 \text{ kips} \leftarrow$$

The relationship between the end forces and any translation such as $E\Delta_a$ is obtained by giving $E\Delta_a$ any value, say 1000, while all rotations θ are kept zero. The fixed-end moments and horizontal reactions in the arch and column are the same as recorded in Example 48. That is, for $E\Delta_a = 1000$ units to the right,

$$M_{Fab} = +8.22 \qquad M_{Fba} = -8.22$$

$$M_{Fae} = M_{Fea} = -15.0 \qquad H_{Fab} = 0.772 = -H_{Fba}$$

These fixed-end moments, Fig. 142*a*, are then distributed in the usual manner and the horizontal end forces computed. The forces F'_a, F'_b, F'_c, and F'_d that are required to make $E\Delta_a = 1000$ while preventing any horizontal motion of b, c, and d are now known. Both the distribution of the end moments and the auxiliary forces are recorded in Fig. 142*a*.

In the same manner the end moments and auxiliary forces F''_a, F''_b, F''_c, and F''_d are obtained in terms of any displacement $E\Delta_b$. These values are recorded in Fig. 142*b*. By symmetry the values for $E\Delta_c$ and $E\Delta_d$ can be obtained by interchanging b and c, also a and d.

To remove the auxiliary forces F when the uniform load is acting on span ab the horizontal translations must be such that

$$\begin{aligned}
1.925\Delta_a - 0.974\Delta_b - 0.178\Delta_c - 0.03\Delta_d &= -F_a = -36.38 \\
-0.974\Delta_a + 2.661\Delta_b - 0.779\Delta_c - 0.178\Delta_d &= -F_b = +33.23 \\
-0.178\Delta_a - 0.779\Delta_b + 2.661\Delta_c - 0.974\Delta_d &= -F_c = +1.86 \\
-0.03\Delta_a - 0.178\Delta_b - 0.974\Delta_c + 1.925\Delta_d &= -F_d = +0.33
\end{aligned}$$

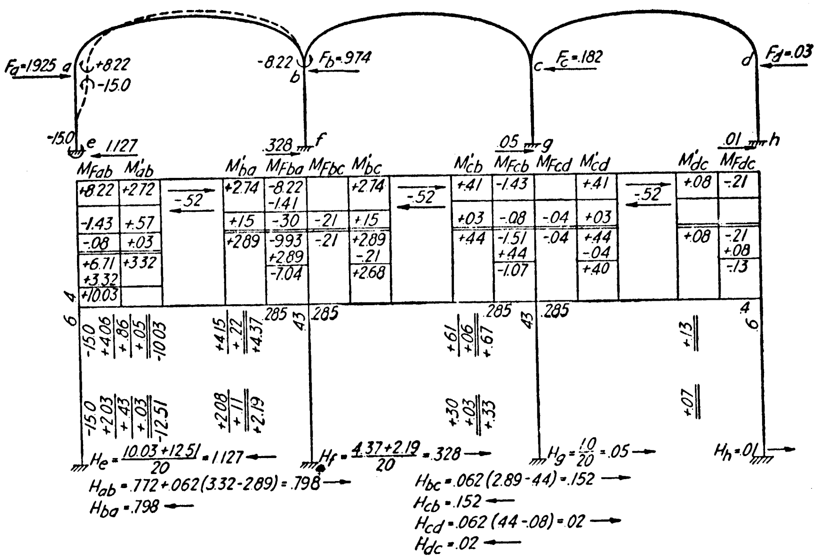


FIG. 142a.

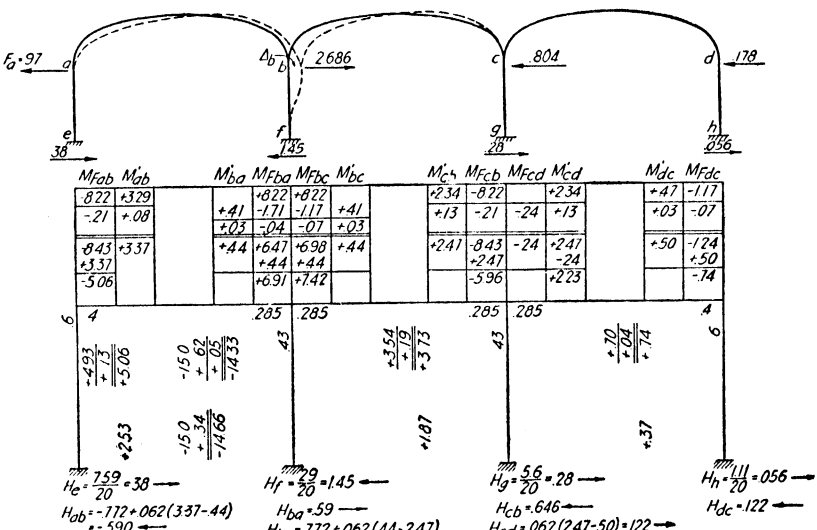


FIG. 142b.

The solution of these equations gives

$$E\Delta_a = -14.5 \qquad E\Delta_b = +8.1 \qquad E\Delta_c = +2.9$$

$$E\Delta_d = +2.2$$

Therefore, the actual moments are equal to

Moments in Fig. 141 + (−14.5) (moments in Fig. 142*a*) + (8.1) (moments in Fig. 142*b*) + (2.9) (moments in Fig. 142*b* reversed) + (2.2) (moments in Fig. 142*a* reversed)

The final moments for the load on span *ab* is given in Fig. 143*a*; Fig. 143*b* gives the values for the load on span *bc*.

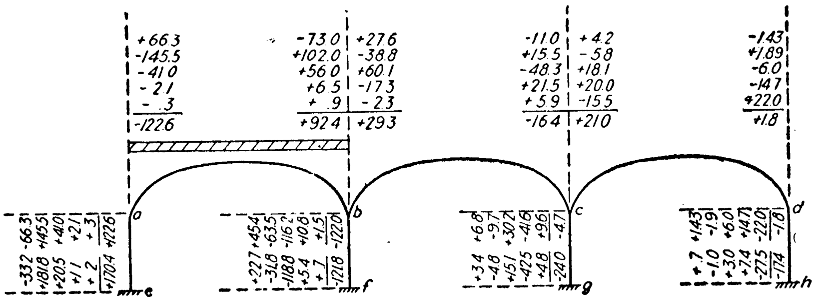


FIG. 143*a*.

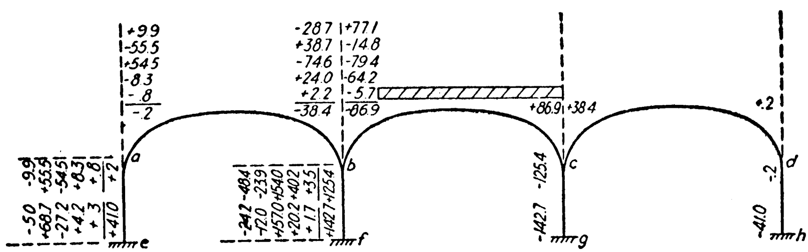


FIG. 143*b*.

80. Elastic Center Method. If the reactions of the fixed-end arch of Fig. 130 are applied at some point O, Fig. 144, whose position is arbitrary, and if the direction of the X' axis, represented by ϕ, is also a variable; then the fundamental equations 116*a*, *b*, and *c*, when applied to the V and X' axes, can be simplified by choosing the proper position of O and value of ϕ. Thus, if point O and angle ϕ are selected so that:

$$\Sigma Gh = 0 \qquad \Sigma Gy' = 0 \qquad \Sigma Ghy' = 0$$

then equations 116a, b, and c reduce to

$$E\Delta_{ox'} = X'_o\,\Sigma Gy'^2 + \Sigma M'Gy' \qquad (125a)$$

$$E\Delta_{oy} = V_o\Sigma Gh^2 - \Sigma M'Gh \qquad (125b)$$

$$E\theta_o = M_o\Sigma G - \Sigma M'G \qquad (125c)$$

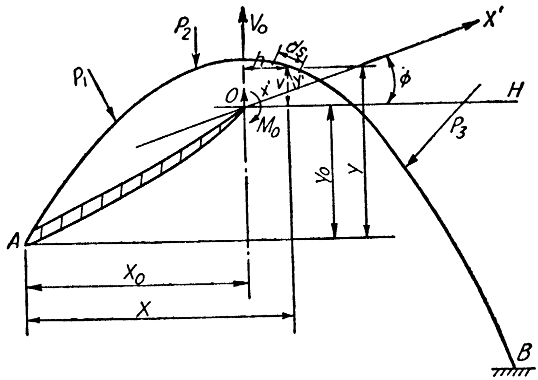


FIG. 144.

Point O is commonly called the "elastic center," as it is located at the centroid of the elastic weights G; and axes X' and V are the principal axes for the G values. In terms of any reference point such as A, Fig. 144, the position of point O as given by the coordinates x_0 and y_0 are

$$x_0 = \frac{\Sigma Gx}{\Sigma G} \qquad y_0 = \frac{\Sigma Gy}{\Sigma G} \qquad (126)$$

To make $\Sigma Ghy' = 0$, let us substitute (see Fig. 145)

$$y' = (v\cos\phi - h\sin\phi)$$

from which

$$\tan\phi = \frac{\Sigma Ghv}{\Sigma Gh^2} \qquad (127)$$

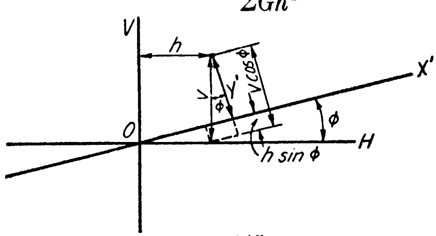


FIG. 145.

For a symmetrical arch or frame $\Sigma Ghv = 0$, and, therefore, the X' axis is horizontal. After the new coordinate system is located, X'_o, V_o,

and M_o can be calculated directly from equations 125a, b, and c, or the graphical solution for influence lines can be used. If an auxiliary force $F_{ox'}$ is applied at point O in the direction of the X' axis, so as to cause a displacement (see equation 125a),

$$E\Delta_{ox'} = F_{ox'}\,\Sigma G y'^2$$

then, from equations 125b and c, the displacements Δ_{oy} and θ_o are zero. As the moment at any element is equal to $F_{ox'}y'$, an influence diagram for the reaction X'_o can be obtained in the usual manner from a funicular polygram constructed by using values of $GF_{ox'}y'$ as loads, and $E\Delta_{ox'}$ as the pole distance of the force diagram. Influence diagrams for V_o and M_o can be constructed in a similar manner by using an auxiliary force F'_{ov} in the direction of V, and a moment M'_o in the direction of M, respectively. The pole distances for these diagrams are $E\Delta_{oy}$ and $E\theta_o$.

Example 51. The frame in Example 49 will be analyzed by means of the elastic center method. The position of the horizontal axis OX' will be obtained by taking moments of the $\dfrac{\Delta s}{I}$ values, or rather the Δs values as I is constant, about a line ab. As each of the eleven elements in the arch has a length of 6.8, Fig. 146,

$$\bar{y} = \frac{6.8\Sigma y - (2)(20)(10)}{(11)(6.8) + (2)(20)} = \frac{(6.8)(117) - 400}{114.8} = 3.44$$

The curved member ab will be divided into the same number of sections as in previous examples, so that

$$v = y' = y - 3.44$$

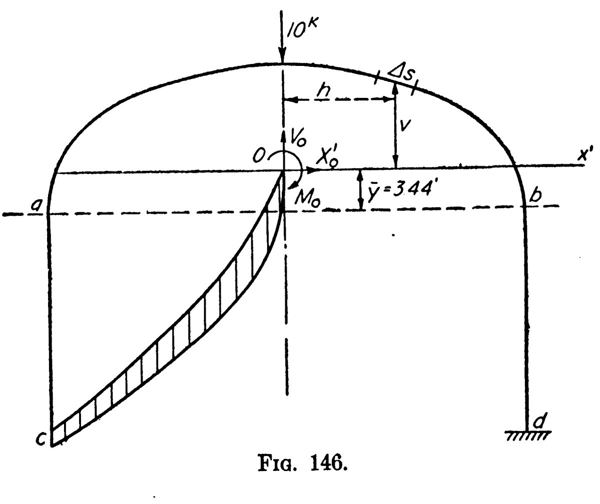


FIG. 146.

From the data in Example 43, the values of M' for the concentrated load of 10 kips at the center of the span and the values of y for the elements of the arch member ab (taking $G = \frac{6.8}{I}$ = unity) determine the following summation.

$$\Sigma M'Gy' = (65.5)(11.16) + (131.0)(10.06) + (193.5)(8.06) + (251.0)(4.76) + (293.0)(-0.24) = 4733$$

For the column bd, the value of $M'Gy'$ equals $\left(\frac{20}{6.8}\right)(300.0)(-13.44) = -11{,}850$

Total $\Sigma M'Gy' = 4733 - 11{,}850 = -7117$

$$\Sigma Gy'^2 = 11.56^2 + 2\left(11.16^2 + 10.06^2 + 8.06^2 + 4.76^2 + (-0.24)^2 + \frac{2}{6.8}\int_{3.44}^{23.44} y^2\,dy\right) = 2022$$

$$\Sigma M'G = 934 + \frac{(300)(20)}{6.8} = 1814$$

$$\Sigma G = 11 + \frac{40}{6.8} = 16.88$$

From equation 125a,

$$X_o'(2022) - 7117 = 0$$

$$X_o' = H = \frac{7117}{2022} = 3.52$$

From equation 125b

$$V_o\Sigma Gh^2 - \Sigma M'Gh = 0$$

but, since $M' = 10h$ for half of the structure,

$$V_o\Sigma Gh^2 - \frac{10\Sigma Gh^2}{2} = 0$$

$$V_o = 5 \text{ kips}$$

From equation 125c

$$M_o(16.88) - 1814 = 0$$

$$M_o = 107.7 \text{ ft-kips}$$

The moment at support c is

$$M_c = 107.7 - (5)(30) + (3.52)(23.44) = 40.2 \text{ ft-kips}$$

81. Analysis of Stiff Rings and Closed Frames. The analysis of stiff rings and closed frames as illustrated in Figs. 147*a* and *b* can be made directly from equations 116*a*, *b*, and *c* as for an arch. In fact, such

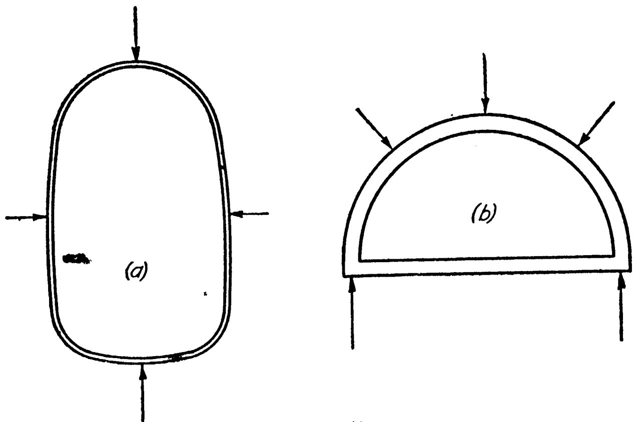


FIG. 147.

structures can be regarded as arches whose abutments coincide. When the equation of the axis of the ring is known, and the moment of inertia is constant, the summations can be replaced by integrals. However, for a general solution adaptable to all conditions, the use of equations 116*a*, *b*, and *c* is recommended. Such structures are frequently symmetrical about the vertical axis, which greatly reduces the numerical work. The signs of the various terms are readily obtained by inspection, and they should be checked before the equations are solved. A numerical example will illustrate the application of the equations to the analysis of stiff rings.

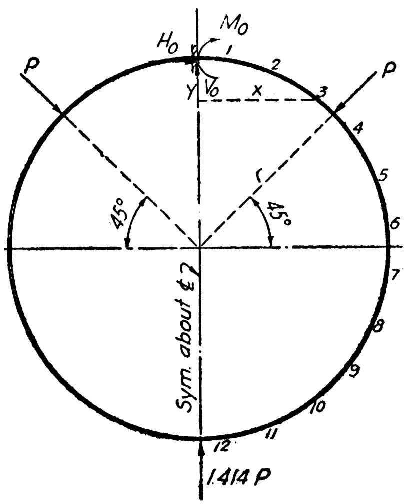


FIG. 148.

Example 52. The circular ring of constant *I* in Fig. 148 is subjected to concentrated loads which are symmetrical with respect to the vertical axis. The ring is cut at the top so that each side can be regarded as a fixed support with equal but opposite forces. Each half of the ring is divided into twelve

equal divisions whose G or $\frac{\Delta s}{I}$ value can be taken as unity. Bending moments that produce tension on the inside of the ring are regarded as positive. Therefore, the unknowns M_0, H_0, and V_0 should be given positive values in the equations and M' negative. Table 10 gives the numerical value of the summations for G and radius r both equal to unity. If the G values are not constant, they must, of course, be included in each term of the summation.

TABLE 10

(For right half of ring; left half is identical except that x is negative)

Section	x	y	x^2	y^2	M'	$M'x$	$M'y$
1	0.1305	0.0086	0.016	.....	0	0	0
2	0.3827	0.0761	0.147	0.005	0	0	0
3	0.6088	0.2066	0.370	0.043	0	0	0
4	0.7934	0.3912	0.630	0.153	−0.1305	−0.104	−0.051
5	0.9239	0.6173	0.854	0.382	−0.3827	−0.353	−0.236
6	0.9914	0.8695	0.983	0.756	−0.6088	−0.604	−0.530
7	0.9914	1.1305	0.983	1.278	−0.7934	−0.787	−0.897
8	0.9239	1.3827	0.854	1.912	−0.9239	−0.854	−1.280
9	0.7934	1.6088	0.630	2.588	−0.9914	−0.787	−1.596
10	0.6088	1.7934	0.370	3.216	−0.9914	−0.604	−1.780
11	0.3827	1.9239	0.147	3.701	−0.9239	−0.353	−1.780
12	0.1305	1.9914	0.016	3.966	−0.7934	−0.104	−1.580
Total		12.00	6.000	18.000	−6.5394	−4.550	−9.730

The values of the summation terms for the entire ring are

$$\Sigma G = 24$$

$$\Sigma Gx = \Sigma Gxy = 0$$

$$\Sigma Gx^2 = 12.0$$

$$\Sigma Gy^2 = 36.0$$

$$\Sigma M'G = -13.0788$$

$$\Sigma M'Gx = 0$$

$$\Sigma M'Gy = -19.46$$

Substituting these values in equations 116a, b, and c, and taking all the unknowns M_0, H_0, and V_0 as positive quantities, gives

$$24.0M_0r + V_0(0)r + 36.0H_0r^2 - 19.46Pr^2 = 0$$

$$0M_0r + 12.0V_0r + (0)(H_0r) - 0 = 0$$

$$24.0M_0 + (0)V_0r + 24.0H_0r - 13.079Pr = 0$$

from which

$$V_0 = 0 \qquad H_0 = 0.532P \qquad M_0 = 0.012Pr$$

The moment at the base is therefore

$$M = 0.012Pr + (2)(0.532Pr) - 0.707Pr = 0.369Pr$$

At the concentrated loads P,

$$M = 0.012Pr + (0.293)(0.532Pr) = 0.168Pr$$

82. Fuselage Frames. In the types of structural framing that are frequently used in aircraft construction the applied loads from wings, tail surfaces, landing gear, water pressure, and various other sources are commonly passed into the hull or fuselage through main frames that are fairly stiff. These applied forces are resisted by shearing stresses between the outer edge of the frame and the skin of the fuselage (Fig. 149). The distribution of these shearing forces is a statically indeterminate problem in itself and is usually solved by means of one of the following assumptions:

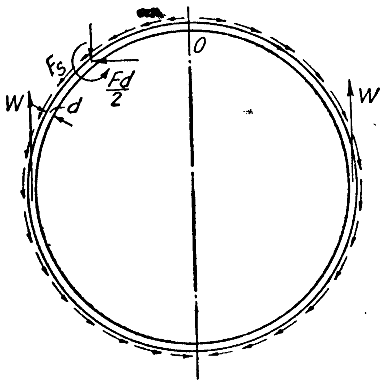


FIG. 149.

(*a*) The shearing stresses are distributed as for any beam section, that is by the formula $q = \dfrac{VQ}{I}$, in which Q and I depend upon the arrangement of effective longitudinal flange area of the fuselage.

(*b*) The shearing stresses are proportional to their distance from the applied load.

In general it would seem that the second assumption will be best if the applied loads are concentrated at one or two points and that the first assumption should be used if the applied load is distributed. As the interaction between the frames and the skin of the fuselage is exceedingly complex, an exact solution is not feasible. Therefore any analysis that is based on an assumed two-dimensional force system must be an approximate solution but should give conservative values for the stresses.

The analysis of any fuselage frame, such as in Fig. 149, is readily made by equations 116 as for a ring. The Δs values, as well as the coordinates of each element, can be obtained graphically if the frame is

drawn to a large scale. For convenience the shearing forces which are tangent to the outer edge can be replaced at the axis of the frame by horizontal and vertical components and a couple $\frac{Fd}{2}$. If the origin is

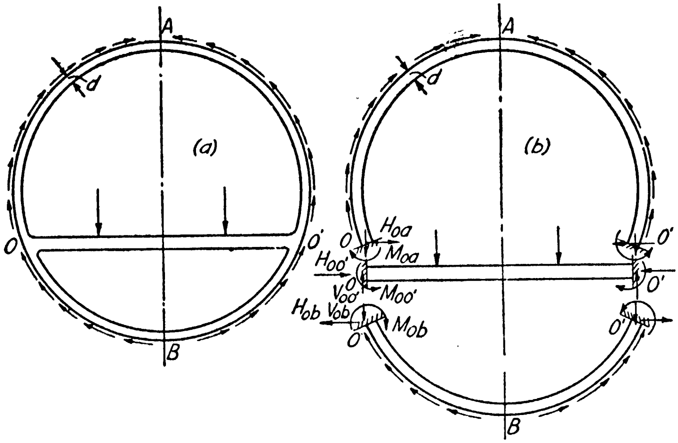


FIG. 150.

taken on the axis at point O at the top of the frame, the solution can be arranged in table form as in Example 52.

If the loads are applied through members that are monolithic with the frame, Fig. 150*a*, the following procedure is recommended:

(*a*) Calculate the fixed-end moments and the horizontal reactions for the separate portions OAO', OBO', and OO' as shown in Fig. 150*b*.

(*b*) Determine the distribution and carry-over factors for each member. (See Articles 78 and 79).

(*c*) Distribute the unbalanced moments and record the necessary corrections to the fixed-end moments and the horizontal reactions.

The analysis will be illustrated by a solution of the frame in Fig. 151.

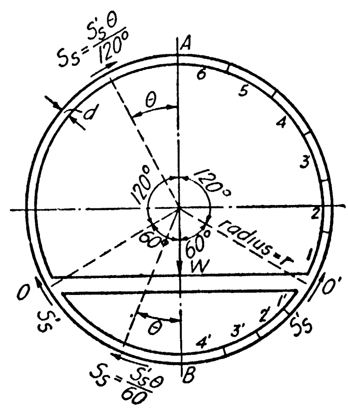


FIG. 151.

Example 53. The calculation of the bending moments at points O and O' (Fig. 151) by the method outlined above requires the magnitude and distribution of the shearing forces along the outer edge of the frame. The second assumption will be used, that is, that the shearing

force per linear foot is proportional to its distance from point O and is zero at the top and bottom of the frame. Accordingly, if q' is the unit shearing force at point O, then q, the unit shearing force at any point, is

$$q = \frac{q'\theta}{120} \text{ for the portion } OA \quad (q = s_s)$$

$$q = \frac{q'\theta}{60} \text{ for the portion } OB \quad (q' = s_s')$$

where θ is measured from A for OA and from B for OB.

The tangential shearing force on any arc Δs will therefore be

$$F_s = q\Delta s = \frac{q'\theta}{120}\Delta s \text{ for portion } OA$$

and

$$F_s = q\Delta s = \frac{q'\theta}{60}\Delta s \text{ for portion } OB$$

The horizontal and vertical components of F_s are

$$F_{sx} = F_s \cos\theta$$

$$F_{sy} = F_s \sin\theta$$

The portion OA will be divided into six equal divisions numbered 1 to 6, and OB into four divisions numbered 1′ to 4′. Table **11** gives the calculations for F_{sx}, F_{sy}, and q'.

TABLE 11

SHEARING FORCE

POINT	θ	$F_s = \frac{\theta}{120} q'\Delta s$	$F_{sx} = F_s \cos\theta$	$F_{sy} = F_s \sin\theta$
6	10°	$\frac{1}{12} q'\Delta s$	$0.082\, q'\Delta s$	$0.0145\, q'\Delta s$
5	30°	$\frac{3}{12}$	0.216	0.1250
4	50°	$\frac{5}{12}$	0.268	0.3192
3	70°	$\frac{7}{12}$	0.199	0.547
2	90°	$\frac{9}{12}$	0	0.750
1	110°	$\frac{11}{12}$	−0.305	0.861
			$\sum_1^6 F_{sx} = 0.460\, q'\Delta s$	$\sum_1^6 F_{sy} = 2.617\, q'\Delta s$
4′	7.5°	$\frac{1}{8}\ q'\Delta s'$	$-0.124\, q'\Delta s'$	$0.0163\, q'\Delta s'$
3′	22.5°	$\frac{3}{8}$	−0.346	0.1438
2′	37.5°	$\frac{5}{8}$	−0.495	0.380
1′	52.5°	$\frac{7}{8}$	−0.533	0.694
			$\sum_{1'}^{4'} F_{sx} = -1.498\, q'\Delta s'$	$\sum_{1'}^{4'} F_{sy} = 1.234\, q'\Delta s'$

From the equilibrium condition

$$\sum_{1}^{6} F_{sy} + \sum_{1'}^{4'} F_{sy} = \frac{W}{2}$$

and

$$\Delta s = \frac{\pi r}{9} \qquad \Delta s' = \frac{\pi r}{12}$$

we obtain

$$(2.617q')\left(\frac{\pi r}{9}\right) + (1.234q')\left(\frac{\pi r}{12}\right) = \frac{W}{2}$$

from which

$$q' = 0.404\frac{W}{r} \qquad q'\Delta s = 0.141W \qquad q'\Delta s' = 0.106W$$

Therefore, all shearing forces and components can be obtained by substituting the values of $q'\Delta s$ and $q'\Delta s'$ in Table 11. If the depth of the frame is d in., the couple acting at each point will be $\frac{F_s d}{2}$, which is shown on the diagrams in Fig. 151. The vertical component at O acting on the portion OAO' is (2.617)(0.141W) or 0.369W, and that on the portion OBO' is (1.234)(0.106W) or 0.131W.

As the vertical reactions are now known they can be considered part of the applied load when computing M' in equations 116a and c, that is,

$$E\Delta_{ox} = -M_{oa}\Sigma Gy + H_{oa}\Sigma Gy^2 - \Sigma M'Gy = 0$$

$$E\theta_{oa} = M_{oa}\Sigma G - H_{oa}\Sigma Gy + \Sigma M'G = 0$$

where M' is the moment at any point with M_{oa} and H_{oa} removed. Only half the structure need be used, as the summations in the above equations for the entire structure are twice that for one half.

TABLE 12. DATA FOR PORTION OAO'

(Origin at O)

Point	x	y	y^2	M'	$M'y$
1	$-0.0737r$	$0.158r$	$0.0250r^2$	$0.0272Wr + 0.065Wd$	$0.0043Wr^2 + 0.010Wdr$
2	$-0.134r$	$0.500r$	$0.2500r^2$	$0.0573Wr + 0.118Wd$	$0.0286Wr^2 + 0.059Wdr$
3	$-0.0737r$	$0.842r$	$0.709r^2$	$0.0639Wr + 0.158Wd$	$0.0538Wr^2 + 0.133Wdr$
4	$+0.1000r$	$1.1428r$	$1.306r^2$	$0.0575Wr + 0.188Wd$	$0.0657Wr^2 + 0.215Wdr$
5	$+0.366r$	$1.366r$	$1.866r^2$	$0.0475Wr + 0.206Wd$	$0.0649Wr^2 + 0.282Wdr$
6	$+0.6924r$	$1.4848r$	$2.205r^2$	$0.0406Wr + 0.211Wd$	$0.0603Wr^2 + 0.313Wdr$
Σ	$0.877r$	$5.4936r$	$6.361r^2$	$0.2940Wr + 0.946Wd$	$0.2776Wr^2 + 1.012Wdr$

After substituting the values from Table 12 into the above equations for $E\Delta_{ox}$ and $E\theta_{oa}$, we obtain (for $G = 1$)

$$-M_{oa}(5.494r) + 6.361r^2H_{oa} = +0.2776Wr^2 + 1.012Wdr$$

$$M_{oa}(6.00) - 5.494rH_{oa} = -0.2940Wr - 0.946Wdr$$

from which

$$H_{oa} = +0.0063W + 0.1095\frac{Wd}{r} \qquad + \rightarrow$$

$$M_{oa} = -0.0432Wr - 0.057Wd \qquad +\circlearrowright$$

In the same manner the reactions at O and O' for the portion OBO' are determined from the summations in Table 13.

TABLE 13. DATA FOR PORTION OBO'

(Origin at O)

Point	x	y	y^2	M'	$M'y$
1′	$0.0726r$	$0.1088r$	$0.0118r^2$	$-0.0095Wr + 0.0066Wd$	$-0.00103Wr^2 + 0.00072Wdr$
2′	$0.2572r$	$0.2934r$	$0.0860r^2$	$-0.0305Wr + 0.0265Wd$	$-0.00894Wr^2 + 0.00776Wdr$
3′	$0.4833r$	$0.4239r$	$0.1800r^2$	$-0.0488Wr + 0.0595Wd$	$-0.02074Wr^2 + 0.02525Wdr$
4′	$0.7355r$	$0.4914r$	$0.2420r^2$	$-0.0591Wr + 0.106Wd$	$-0.02910Wr^2 + 0.05215Wdr$
Σ		$1.3175r$	$0.5198r^2$	$-0.1479Wr + 0.1986Wd$	$-0.05981Wr^2 + 0.08588Wdr$

$$1.318M_{ob}r + 0.520H_{ob}r^2 = 0.0598Wr^2 - 0.0859Wdr$$

$$4.00M_{ob} + 1.318H_{ob}r = 0.1479Wr - 0.1986Wd$$

from which

$$H_{ob} = 0.1293W - 0.237\frac{Wd}{r} \qquad + \rightarrow$$

$$M_{ob} = -0.0056Wr + 0.0285Wd \qquad +\circlearrowright$$

For the beam OO' the fixed-end moments for a concentrated load W at the center is

$$M_{Foo'} = -\frac{WL}{8} = -\frac{W2r\sin 60°}{8} = -0.216Wr$$

As the fixed-end forces at the joints O and O' have now been calculated for all three members, there remains only the task of distributing the unbalanced moment to each member. This problem can be solved in the usual way by rotating the joints O and O' separately, but the easiest procedure is to take advantage of symmetry and rotate both joints at the same time and thereby eliminate the carry-over procedure. The correction is then made in one operation.

If two equal and opposite moments M'_o are applied to a symmetrical curved member OAO', Fig. 152a, and the ends are restrained against horizontal displacement, then, from equations 116a and c,

$$E\Delta_o = -M'_o\Sigma Gy + H'_o\Sigma Gy^2 = 0$$

from which

$$H'_o = \frac{\Sigma Gy}{\Sigma Gy^2} M'_o \qquad (128a)$$

and

$$2E\theta_o = M'_o\Sigma G - H'_o\Sigma Gy$$

or

$$M'_o = \frac{2E\theta_o}{\Sigma G - \dfrac{(\Sigma Gy)^2}{\Sigma Gy^2}} \qquad (128b)$$

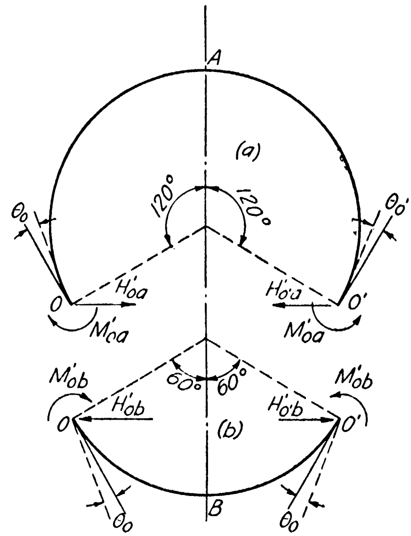


FIG. 152.

Since G represents $\dfrac{\Delta s}{I}$ for each element, the above equations can be used for frames with variable I. For the portion OAO', G equals a constant $\dfrac{\pi r}{9I}$ for all elements and therefore (values from Table 11)

$$M'_{oa} = \frac{2E\theta_o}{(2)\left(\dfrac{\pi r}{9I}\right)\left[6.0 - \dfrac{(5.494)^2}{6.361}\right]} = 2.26\frac{EI}{r}\theta_o$$

and

$$H'_{oa} = \frac{5.494}{6.361}\frac{M'_{oa}}{r} = 0.863\frac{M'_{oa}}{r}$$

acting to the right for clockwise moment.

For the portion OBO',

$$M'_{ob} = \frac{2E\theta_o}{(2)\left(\dfrac{\pi r}{12I}\right)\left[4.0 - \dfrac{(1.318)^2}{0.520}\right]} = 5.78\frac{EI}{r}\theta_o$$

and

$$H'_{ob} = \frac{1.318}{0.520}\frac{M'_{ob}}{r} = 2.53\frac{M^o_{ob}}{r}$$

acting to the left for clockwise moment.

For the beam OO', the relations are

$$M_{oo'} = \frac{EI}{L}[4\theta_o + 2\theta_o'] = \frac{2EI}{L}\theta_o$$

or, since

$$L = (2)(0.866r)$$

$$M_{oo'} = 1.15\frac{EI}{r}\theta_o$$

If the rotation θ_o is assumed to be the same for both frame and beam, the distribution factors are

For OAO'

$$\frac{2.26I_f}{2.26I_f + 5.78I_f + 1.15I_b}$$

where I_f = moment of inertia of frame.
I_b = moment of inertia of beam.

For OBO'

$$\frac{5.78I_f}{2.26I_f + 5.78I_f + 1.15I_b}$$

For OO'

$$\frac{1.15I_b}{2.26I_f + 5.78I_f + 1.15I_b}$$

If we let $I_b = 10I_f$, the distribution factors are

For OAO'	0.116
For OBO'	0.296
For OO'	0.588
	1.000

Let $r = 60$ in., $d = 5$ in.
Fixed-end moments are

$$M_{Foa} = (-0.0432W)(60) - (0.057W)(5) = -2.592W - 0.285W$$
$$= -2.867W \text{ in.-lb}$$

$$M_{Fob} = -0.0056W(60) + (0.0285W)(5) = -0.194W \text{ in.-lb}$$

$$M_{Foo'} = -0.216W(60) = -12.96W$$

$$\Sigma M_{Fo} = -2.87W - 0.19W - 12.96W = -16.02W$$

Therefore, the actual end moments are

$$M_{oa} = -2.87W + (0.116)(16.02W) = -1.01W$$

$$M_{ob} = -0.19W + (0.296)(16.02W) = +4.55W$$

$$M_{oo'} = -12.96W + (0.588)(16.02W) = -3.54W$$

The final values of H are

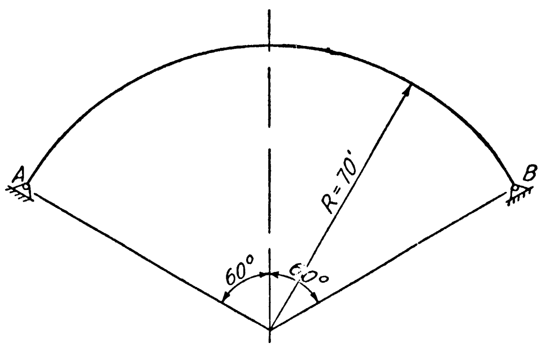
$$H_{oa} = +0.0063W + (0.1095W)\frac{5}{60} + \frac{(0.863)(1.86W)}{60} = 0.0322W \rightarrow$$

$$H_{ob} = +0.1293W - (0.237W)\frac{5}{60} - \frac{(2.53)(4.74W)}{60} = -0.0905W \leftarrow$$

$$H_{oo'} = +0.0905W - 0.0322W = 0.0583W \rightarrow$$

PROBLEMS

67. The two-hinged steel arch shown has a symmetrical cross section with area of 24 in.^{2}, a depth of 18 in., and a moment of inertia of 2400 in.^{4}



PROBLEM 67.

(a) Construct an influence diagram for the horizontal reaction, H, by the graphical method. Check the ordinate at the center of the span by an algebraic solution.

(b) Construct influence diagrams for the thrust and bending moment at the quarter point and center of span.

(c) Calculate the maximum unit stress at the quarter point and center of span for a dead load of 800 lb per linear foot; and for a live load of 1200 lb per linear foot.

68. Calculate the unit stress at the quarter point and at the center of the span of the arch in Problem 67 for a temperature change of $\pm 60°$ F. Use a coefficient of linear expansion of 6.5×10^{-6}, and a modulus of elasticity of 29.5×10^6 lb per in.^{2}

69. What is the vertical deflection at the center of the span of the steel arch in Problem 67 for the combined weight of 2000 lb per linear foot across the entire span and a temperature drop of 60° F?

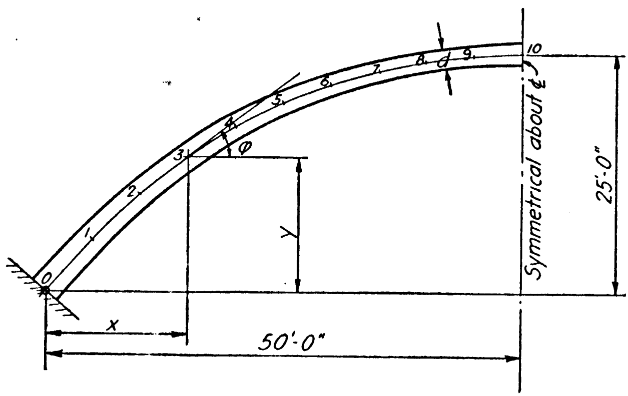
70. If the steel arch of Problem 67 is fixed at the supports instead of hinged, solve the following problems by the methods illustrated in the text.

(a) Construct influence diagrams for the horizontal and vertical components (H_a, V_a), and the moment M_a at the support.

(b) Construct influence diagrams for the thrust and bending moment at the quarter point and center of span.

(c) Calculate the horizontal component H_a and moment M_a for a temperature change of $\pm 60°$ F for the coefficients given in Problem 68.

71. The reinforced-concrete arch shown is fixed at the abutments and has the dimensions given in the table below. Construct influence diagrams for the



PROBLEM 71.

reactions H_a, V_a, and M_a by the semi-graphical method. Assume that the width is constant and that the moment of inertia varies as the cube of the depth.

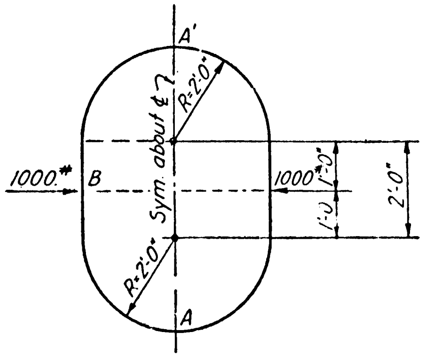
POINT	x (ft)	y (ft)	DEPTH (in.)	TAN ϕ
0	0	0.	42.0	1.229
1	5	5.61	37.7	1.023
2	10	10.27	34.6	0.846
3	15	14.11	32.2	0.694
4	20	17.24	30.1	0.561
5	25	19.75	28.6	0.445
6	30	21.71	27.2	0.342
7	35	23.18	26.2	0.248
8	40	24.20	25.4	0.162
9	45	24.80	24.7	0.080
10	50	25.00	24.0	0.0

72. Determine the position of the elastic center of the arch in Problem 71. Calculate the reactions H_a, V_a, and M_a for a concentrated load of 10 kips at the

center of the span by use of the elastic center. Check with the values that were obtained in Problem 71.

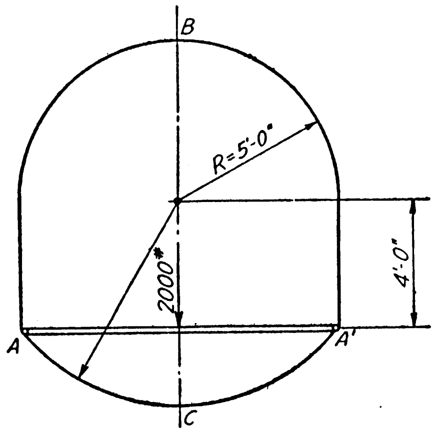
73. Determine the stiffness factor, carry-over factor, and correction for H_a for the arch in Problem 71, from the numerical values used in the solution of that problem.

74. What are the internal forces acting at points A and B in the frame shown?



PROBLEM 74.

75. Determine the internal forces acting at points A and A' of the fuselage frame shown. Assume that the resisting shear forces vary from a maximum



PROBLEM 75.

at A to zero at points B and C, as in Example 53, and that they are acting on the axis of the frame. The frame $ABA'C$ is continuous, but the member AA' is hinged.

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CHAPTER IX

FLEXIBLE MEMBERS

83. Introduction. In the preceding chapters various structures have been analyzed for the condition that the deflection is sufficiently small, as compared to the overall dimensions, to be neglected when calculating stresses and displacements. As the length-depth ratios of the members increase, however, proportions are reached where such an assumption is not permissible. The error involved in neglecting the displacement of the axes of the members in flexible structures may lead either to dangerous conclusions or to an uneconomical design. It therefore seems essential that the engineer be familiar with such structures, at least to the extent of recognizing the problems when they occur. At the same time, it will often be expedient to avoid the mathematically complex exact solutions by substituting a solution by successive approximations or even an approximate one. The fundamental equations must, of course, always remain the basis for judging the validity of any short cut in the numerical calculations. A number of

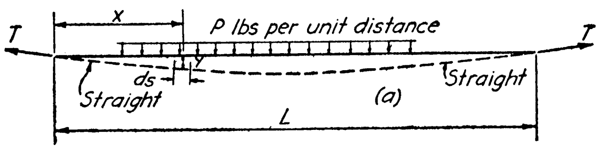


FIG. 153*a*.

statically indeterminate problems in which the deflection is important will now be considered to illustrate both the fundamental equations and the use of successive approximations.

84. Wires Subjected to Radial Pressure. When a straight member that has no flexural resistance is subjected to a radial pressure, Fig. 153*a*, the tension T must be constant throughout. The tension is constant because the equilibrium of any element ds, Fig. 153*b*, requires that

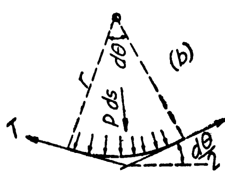


FIG. 153*b*.

$$T_1 \cos\frac{d\theta}{2} = T_2 \cos\frac{d\theta}{2} \quad \therefore \quad T_1 = T_2 = T$$

and

$$\frac{p\,ds}{2T} = \sin\frac{d\theta}{2} = \frac{d\theta}{2} \quad \text{(for small angles)} \tag{129a}$$

Since

$$ds = r\,d\theta$$

$$p = \frac{T}{r} = T\frac{d^2y}{dx^2} \tag{129b}$$

as

$$\frac{1}{r} = \frac{d^2y}{dx^2} \quad \text{for small curvature}$$

Although the above equation of equilibrium gives the relation between T, y, p, and L, the numerical value of T cannot be calculated directly. To determine T it is necessary to combine the above equilibrium equation with the strain condition that the change in length of the wire due to the constant tension T be consistent with the displacements necessary to provide equilibrium. The change in length of the wire ΔL due to the tension T is

$$\Delta L = \frac{TL}{AE} \tag{130a}$$

while the change in length due to relatively small deflections is obtained in the following manner:

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Expanding by the binomial theorem gives

$$ds = dx\left[1 + \tfrac{1}{2}\left(\frac{dy}{dx}\right)^2 - \tfrac{1}{8}\left(\frac{dy}{dx}\right)^4 + \cdots\right]$$

If only the first two terms of the series are used, the increase in length equals

$$\Delta L = \int_0^L ds - \int_0^L dx = \frac{1}{2}\int_0^L \left(\frac{dy}{dx}\right)^2 dx \tag{130b}$$

Equating this value to that obtained by equation 130a gives

$$\frac{TL}{AE} = \frac{1}{2}\int_0^L \left(\frac{dy}{dx}\right)^2 dx \tag{131}$$

From equation 129

$$\frac{dy}{dx} = \frac{1}{T}\left[\int p\,dx + C_1\right] \tag{132}$$

so that a numerical solution for T can be made for any distribution of p.

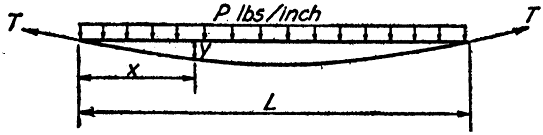


FIG. 154.

Example 54. The value of T for a constant pressure p over the entire span, Fig. 154, will be calculated.

$$T\frac{d^2y}{dx^2} = p$$

$$T\frac{dy}{dx} = px + C_1$$

$$Ty = \frac{px^2}{2} + C_1x + C_2$$

When

$$x = 0 \qquad y = 0 \qquad C_2 = 0$$

$$x = L \qquad y = 0 \qquad C_1 = -\frac{pL}{2}$$

Therefore,

$$\frac{dy}{dx} = \frac{p}{T}\left(x - \frac{L}{2}\right)$$

Substituting this expression in equation 131 gives

$$\frac{TL}{AE} = \frac{1}{2}\int_0^L \frac{p^2}{T^2}\left(x - \frac{L}{2}\right)^2 dx$$

from which

$$T^3 = \frac{p^2L^2AE}{24}$$

If $p = 4$ lb per in., $L = 20$ ft, $A = 0.05$ in.^{2}, and $E = 27 \times 10^6$ lb per in.^{2},

$$T^3 = \frac{(4)(4)(400)(144)(0.05)(27)(10^6)}{24} = (52)(10^9)$$

$$T = 1000\sqrt[3]{52} = 3730 \text{ lb}$$

or a unit stress of $\dfrac{3730}{0.05} = 74{,}600$ lb per in.^{2}

If the wire has an initial tension T_0 before the pressure is applied, the change in length ΔL is due to the difference between the initial and final tension, or

$$\Delta L = \frac{(T - T_0)L}{AE} = \frac{1}{2}\int_0^L \left(\frac{dy}{dx}\right)^2 dx$$

For a constant pressure p and initial tension T_0, we obtain

$$\frac{(T - T_0)L}{AE} = \frac{1}{2}\int_0^L \frac{p^2}{T^2}\left(x - \frac{L}{2}\right)^2 dx$$

from which

$$T^2(T - T_0) = \frac{p^2L^2AE}{24}$$

or

$$T = \frac{pL}{2}\sqrt{\frac{AE}{6(T - T_0)}}$$

This equation can be quickly solved by trial.

Let T_0 equal 800 lb with the other values remaining as before. Then

$$\frac{pL}{2} = \frac{(4)(20)(12)}{2} = 480 \qquad AE = 135 \times 10^4$$

or

$$T = 48{,}000\sqrt{\frac{135}{6(T - 800)}}$$

By trial, $T = 4010$ lb. Any change in span length due to temperature or movement of the supports can be added algebraically to ΔL.

85. Compression Members with Transverse Loads. When a flexible member with some flexural strength is subjected to combined axial

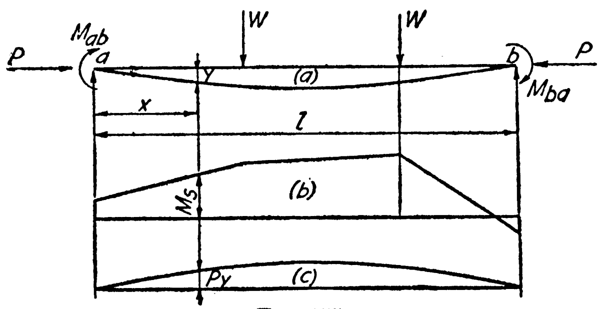


FIG. 155.

and transverse loads as in Fig. 155a, the bending moments that are caused by the axial load P acting through the displacement y may be

of considerable magnitude. As the calculation of the stresses in such members is fully discussed in many books on strength of materials, only the deformation equations that are necessary for the analysis of continuous beams and frames will be treated here. In the following derivation it is assumed that the stresses are within the proportional limit and that the member has sufficient rigidity to prevent buckling.

The bending moment M_x at any section of the member ab, Fig. 155, is equal to

$$M_x = M_s + Py \tag{133}$$

where M_s is the bending moment when the axial load P is not acting, Fig. 155*b*. The equation of the elastic curve is

$$\frac{d^2y}{dx^2} = -\frac{M_x}{EI} \tag{134}$$

and, for a member with constant EI, this equation becomes

$$\frac{d^2y}{dx^2} + a^2y = -a^2\left(\frac{M_s}{P}\right) \tag{135}$$

in which $a^2 = \dfrac{P}{EI}$. The general solution of this equation is

$$y = A\cos ax + B\sin ax - \frac{M_s}{P} + \frac{1}{a^2}\frac{M_s''}{P} - \frac{1}{a^4}\frac{M_s^{\mathrm{IV}}}{P} + \cdots \tag{136}$$

in which

$$M_s'' = \frac{d^2M_s}{dx^2} \qquad M^{\mathrm{IV}} = \frac{d^4M_s}{dx^4}$$

For a uniform load of w pounds per unit length, and end couples of M_{ab} and M_{ba},

$$M_s = \frac{M_{ab}(l-x)}{l} - M_{ba}\frac{x}{l} + \frac{wl}{2}x - \frac{wx^2}{2}$$

$$M_s'' = -w \qquad M_s^{\mathrm{IV}} = 0$$

or

$$y = A\cos ax + B\sin ax$$

$$-\frac{M_{ab}(l-x)}{Pl} + \frac{M_{ba}x}{Pl} - \frac{wl}{2}\frac{x}{P} + \frac{wx^2}{2P} - \frac{w}{a^2P} \tag{137}$$

The boundary conditions are

$$y = 0 \qquad x = 0 \qquad \text{or} \qquad A = \frac{M_{ab}}{P} + \frac{w}{a^2P} \tag{138a}$$

$$y = 0 \qquad x = l \qquad \text{or}$$

$$B = \frac{\left(\frac{M_{ab}}{P} + \frac{w}{a^2P}\right)\cos al - \frac{M_{ba}}{P} + \frac{w}{a^2P}}{\sin al} \tag{138b}$$

As the slope at any point of the elastic curve can be obtained from the equation

$$\frac{dy}{dx} = -Aa\sin ax + Ba\cos ax - \frac{M_s'}{P} + \frac{1}{a^2}\frac{M_s'''}{P} \tag{139a}$$

the end rotations θ_a and θ_b can be determined by substituting the proper value of x, that is, for $x = 0$

$$\theta_a = \left(\frac{dy}{dx}\right)_{x=0} = Ba - \frac{M_s'}{P} \tag{139b}$$

After substituting the value of B and M_s' for $x = 0$ in equation 139b and replacing al by α, we obtain the following relation:

$$\theta_a = \frac{M_{ab}}{Pl}(1 - \alpha\cot\alpha) - \frac{M_{ba}}{Pl}(\alpha\operatorname{cosec}\alpha - 1) + \frac{wl}{P}\left(\frac{\tan\alpha/2}{\alpha} - \frac{1}{2}\right) \tag{140a}$$

In the same manner, for $x = l$,

$$\theta_b = \left(\frac{dy}{dx}\right)_{x=l} = -\frac{M_{ab}}{Pl}(\alpha\operatorname{cosec}\alpha - 1) + \frac{M_{ba}}{Pl}(1 - \alpha\cot\alpha) - \frac{wl}{P}\left(\frac{\tan\alpha/2}{\alpha} - \frac{1}{2}\right) \tag{140b}$$

in which $\alpha = \sqrt{\frac{Pl^2}{EI}}$. Solving equation 140$a$ and b for M_{ab} and M_{ba} gives the usual form of the slope-deflection equations

$$M_{ab} = C_1K\theta_a + C_2K\theta_b + (C_1 + C_2)K\frac{\Delta}{L} + M_{Fab} \tag{141a}$$

$$M_{ba} = C_2K\theta_a + C_1K\theta_b + (C_1 + C_2)K\frac{\Delta}{L} + M_{Fba} \tag{141b}$$

in which

$$C_1 = \frac{1 - \alpha \cot \alpha}{\dfrac{2 \tan \alpha/2}{\alpha} - 1} \tag{142a}$$

$$C_2 = \frac{\alpha \operatorname{cosec} \alpha - 1}{\dfrac{2 \tan \alpha/2}{\alpha} - 1} \tag{142b}$$

$$M_{Fab} = -M_{Fba} = \left(\frac{1 - \dfrac{2 \tan \alpha/2}{\alpha}}{2\alpha \tan \alpha/2}\right) wl^2 \tag{142c}$$

$$K = \frac{EI}{l} \qquad \alpha = \sqrt{\frac{Pl^2}{EI}}$$

After the values of C_1K, C_2K, and the fixed-end moments are determined, the distribution and carry-over factors are calculated in the usual manner. The carry-over factor is, of course,

$$\frac{C_2}{C_1} = \frac{\alpha \operatorname{cosec} \alpha - 1}{1 - \alpha \cot \alpha} \tag{142d}$$

A diagram in the Appendix gives C_1 and $\dfrac{C_2}{C_1}$ values for various values of α.

86. Tension Members with Transverse Loads. If an axial tensile load instead of compression is used, Fig. 156, the equations become

$$M_x = M_s - Py \tag{143a}$$

$$\frac{d^2y}{dx^2} - a^2y = -a^2\left(\frac{M_s}{P}\right) \tag{143b}$$

in which $a^2 = \dfrac{P}{EI}$.

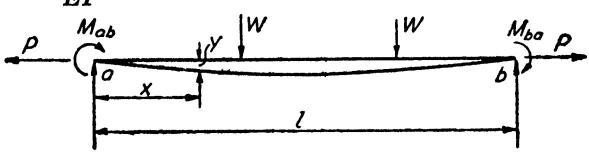


FIG. 156.

The general solution of this differential equation gives the equation of the elastic curve as

$$y = A \cosh ax + B \sinh ax + \frac{M_s}{P} + \frac{1}{a^2}\frac{M_s''}{P} + \frac{1}{a^4}\frac{M_s^{\text{IV}}}{P} \tag{143c}$$

If the evaluation of the constants A and B and the end slopes θ_a and θ_b is carried out for a uniform distribution w pounds per unit length and for end couples M_{ab} and M_{ba} in the same manner as was previously used for the axial compressive force, we again obtain

$$M_{ab} = C_1 K\theta_a + C_2 K\theta_b + (C_1 + C_2)K\frac{\Delta}{L} + M_{Fab} \qquad (141a)$$

$$M_{ba} = C_2 K\theta_a + C_1 K\theta_b + (C_1 + C_2)K\frac{\Delta}{L} + M_{Fba} \qquad (141b)$$

in which

$$C_1 = \frac{\alpha \coth \alpha - 1}{1 - \dfrac{2 \tanh \alpha/2}{\alpha}} \qquad (144a)$$

$$C_2 = \frac{1 - \alpha \operatorname{cosch} \alpha}{1 - \dfrac{2 \tanh \alpha/2}{\alpha}} \qquad (144b)$$

$$M_{Fab} = -M_{Fba} = \left(\frac{2 \tanh \alpha/2 - \alpha}{2\alpha^2 \tanh \alpha/2}\right) wl^2 \qquad (144c)$$

$$\alpha = \sqrt{\frac{Pl^2}{EI}}$$

$$K = \frac{EI}{l}$$

Values of C_1 and $\dfrac{C_2}{C_1}$ for various values of α can be obtained from diagrams in the Appendix.

87. Principle of Superposition. Reciprocal Theorem. From equations 136 and 143 it is apparent that the displacements y and θ are linear functions of M_{ab}, M_{ba}, and W if P, E, I, and l are kept constant. Consequently, the displacements due to any combination of transverse loads can be added algebraically in any order provided that all displacements are calculated with the same value of the axial load P acting. However, if the axial load varies with the transverse loading, as it frequently does, the true value of P may be difficult to obtain. Fortunately, the value of the axial load P in most structures is not sensitive to changes in the stiffness of the member.

If the development of the reciprocal theorem in Article 7, Chapter II, is studied it will be found that the discussion there will apply to members with both transverse and axial loads if, in addition to the work

done by the vertical loads, there is added an amount of work done by the axial load P equal to

$$P\,\Delta L = \frac{P}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx \tag{145}$$

However, compared to the $\frac{1}{2}P\Delta$ terms in equation 6 the work indicated by equation 145 can be neglected for an originally straight member with small displacements. As we shall see later, this assumption is sufficiently accurate for many practical problems even when the structure is originally curved, like an arch or suspension bridge. In general it is safe to use the reciprocal theorem when the α term is kept constant.

Example 55. The fixed-end moments for the member ab in Fig. 157a will be calculated for a value of α equal to 2 by means of the reciprocal theorem. If any moment M_{ab} is applied to the member as shown in Fig. 157b, producing the elastic curve shown, then, by the reciprocal theorem,

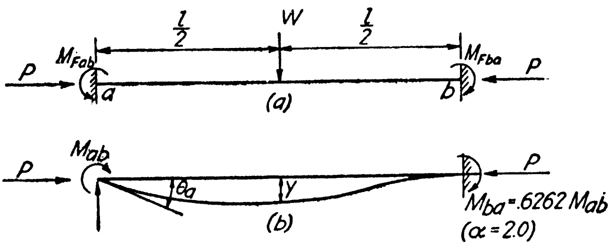


FIG. 157.

$$M_{Fab} = W\left(\frac{y}{\theta_a}\right)$$

From equation 142d, the value of M_{ba} is

$$M_{ba} = \frac{C_2}{C_1}M_{ab} = \frac{\alpha \operatorname{cosec}\alpha - 1}{1 - \alpha\cot\alpha}M_{ab} = 0.6262M_{ab}$$

From equations 137 and 138

$$(y)_{x=l/2} = \frac{M_{ab}}{P}\cos\frac{\alpha}{2} - \frac{M_{ab}}{P}\cot\alpha\sin\frac{\alpha}{2} - \frac{M_{ba}}{P}\operatorname{cosec}\alpha\sin\frac{\alpha}{2} - \frac{M_{ab}}{2P} + \frac{M_{ba}}{2P}$$

which gives for $\alpha = 2$

$$(y)_{x=l/2} = 0.159\frac{M_{ab}}{P}$$

From equation 140a,

$$\theta_a = \frac{M_{ab}}{Pl}(1 - \alpha \cot \alpha) - \frac{M_{ba}}{Pl}(\alpha \operatorname{cosec} \alpha - 1)$$

from which

$$\theta_a = 1.1642 \frac{M_{ab}}{Pl}$$

Therefore,

$$M_{Fab} = W\left(\frac{y}{\theta_a}\right) = W\left(\frac{0.159 \dfrac{M_{ab}}{P}}{1.1642 \dfrac{M_{ab}}{Pl}}\right)$$

$$= 0.1365Wl \quad (.1361Wl \text{ by exact solution})$$

The fixed-end moments can be calculated by the above method for any position of W and for any value of α.

88. Use of Trigonometric Series. The mathematical difficulties that are involved in the solution of equation 136 may often be avoided by the use of trigonometric series in calculating the displacements y and θ. In equations 133 and 134, the bending moment M_s and displacement y can usually be represented by a Fourier series of the type

$$M_s = \sum_{n=1}^{\infty} A'_n \sin \frac{n\pi x}{l} \tag{146a}$$

$$y = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \tag{146b}$$

and from equation 146b we obtain

$$\frac{d^2y}{dx^2} = -\frac{\pi^2}{l^2} \sum_{n=1}^{\infty} n^2 A_n \sin \frac{n\pi x}{l}$$

Substituting these values in equation 134 gives

$$-EI \frac{\pi^2}{l^2} \sum_{n=1}^{\infty} n^2 A_n \sin \frac{n\pi x}{l} = -\sum_{n=1}^{\infty} A'_n \sin \frac{n\pi x}{l} - P \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \tag{147a}$$

To satisfy equation 147a for each term of the series, the coefficients must have the relation

$$A_n = \frac{l^2}{EI}\left(\frac{A'_n}{n^2\pi^2 - \alpha^2}\right) \tag{147b}$$

The elastic curve is therefore known whenever the coefficients A'_n for the bending moments M_s are determined. These coefficients can ordinarily be calculated from the requirement that

$$A'_n = \frac{2}{l}\int_0^l M_s \sin\frac{n\pi x}{l}\,dx \tag{147c}$$

For example, if a concentrated load W is applied at a distance c from one end, the equations for M_s are

$$M_s = \frac{W(l-c)}{l}x \qquad \text{for } x \to 0 \text{ to } c$$

$$M_s = \frac{Wc}{l}(l-x) \qquad \text{for } x \to c \text{ to } l$$

From equation 147c, we obtain

$$A'_n = \frac{2}{l}\left[\int_0^c \frac{W(l-c)}{l}x\sin\frac{n\pi x}{l}\,dx + \int_c^l \frac{Wc(l-x)}{l}\sin\frac{n\pi x}{l}\,dx\right]$$

from which

$$A'_n = \frac{2Wl}{n^2\pi^2}\sin\frac{n\pi c}{l} \tag{148a}$$

and the equation of the elastic curve as given by equations 146b and 147b is

$$A_n = \frac{2Wl^3}{\pi^2 EI}\left[\frac{\sin\frac{n\pi c}{l}}{n^2(n^2\pi^2-\alpha^2)}\right] \tag{148b}$$

$$y = \frac{2Wl^3}{\pi^2 EI}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi c}{l}\sin\frac{n\pi x}{l}}{n^2(n^2\pi^2-\alpha^2)} \tag{148c}$$

$$\frac{dy}{dx} = \frac{2Wl^2}{\pi EI}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi c}{l}\cos\frac{n\pi x}{l}}{n(n^2\pi^2-\alpha^2)} \tag{148d}$$

The end slopes are readily obtained from equation 148d, and, with the end rotations known, the fixed-end moments can be calculated from equations 141a and b in the same manner as for beams without axial loads. By changing the term $n^2\pi^2 - \alpha^2$ in the denominator of equation 147b to $n^2\pi^2 + \alpha^2$, equations 147 and 148 can also be used when an axial tensile force is acting.

Example 56. The fixed-end moments for the member in Fig. 157a will be calculated from equations 141a and b by taking θ_a and θ_b equal to the rotations due to the transverse load W but with opposite sign.

From equation 148d, for

$$x = 0 \qquad c = \frac{l}{2}$$

$$\theta_a = -\theta_b = \frac{2Wl^2}{\pi EI}\left[\frac{1}{\pi^2 - \alpha^2} - \frac{1}{3(9\pi^2 - \alpha^2)} + \frac{1}{5(25\pi^2 - \alpha^2)} \cdots\right]$$

For

$$\alpha = 2 \qquad \frac{EI}{L} = K$$

$$\theta_a = -\theta_b = \frac{2Wl}{\pi K}[0.1703 - 0.0039] = 0.1059\frac{Wl}{K}$$

Using equations 142a and b, we obtain

$$C_1 = \frac{1 - 2\cot 2}{\dfrac{2\tan 1}{2} - 1} = \frac{1.9153}{0.55741} = 3.435$$

$$C_2 = \frac{2\operatorname{cosec} 2 - 1}{\dfrac{2\tan 1}{2} - 1} = \frac{1.1995}{0.55741} = 2.15$$

and from equation 141a,

$$M_{Fab} = 3.435K\left(-0.1059\frac{Wl}{K}\right) + 2.15K\left(0.1059\frac{Wl}{K}\right) = -0.1361Wl$$

89. Continuous Beams with Axial Loads. After the coefficients and fixed-end moments for each span of a continuous beam with axial loads have been calculated by equation 142 or 144, the actual end moments can then be obtained by the moment-distribution method in the usual manner. For axial compressive loads the carry-over factors are larger than 0.5, and as these carry-over factors increase the rate of convergence of the corrections decreases. When the value of α is equal to π the carry-over factor is equal to unity. For this value the convergence by the moment-distribution method is likely to be slow and, therefore, for such problems the direct use of equations 141a and b as in the slope-deflection solution in Chapter III is preferable. Both the moment-distribution and slope-deflection methods are illustrated in Example 57.

Example 57. The end moments for the continuous beam shown in Fig. 158 will be calculated by both the moment-distribution and slope-deflection methods.

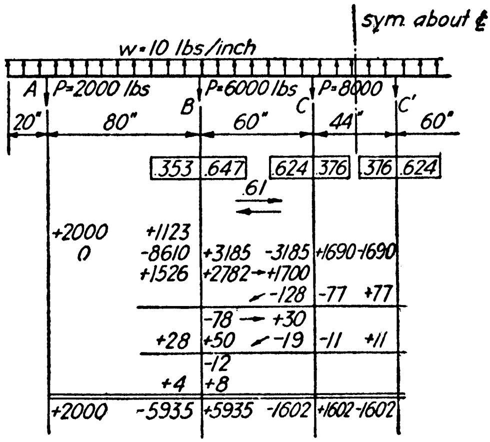


FIG. 158.

For all spans

$$E = 1.5 \times 10^6 \text{ lb per in.}^2$$

$$I = 4.0 \text{ in.}^4$$

$$EI = 6.0 \times 10^6 \text{ in.}^2\text{-lb}$$

For span AB

$$\alpha = \sqrt{\frac{(2000)(80)(80)}{(6.0)(10^6)}} = 1.46$$

from equations 142a and b,

$$C_1 = \frac{1 - 0.163}{1.225 - 1} = 3.72$$

$$C_2 = \frac{\dfrac{1.46}{0.9939} - 1}{0.225} = 2.09$$

$$C_b' = C_1 - \frac{C_2^2}{C_1} = 2.55 \quad \text{(For hinge at } a)$$

$$M_{FAB} = \frac{1 - 1.225}{(2)(1.46)(.8949)} wl^2 = (-0.0861)(-10)(80)^2 = 5510 \text{ in.-lb}$$

$$M_{FBA} = -5510 \text{ in.-lb}$$

For a hinge at A

$$M_{FBA} = -5510 - \left(\frac{2.09}{3.72}\right)(5510) = -8610 \text{ in.-lb}$$

For span BC

$$\alpha = \sqrt{\frac{(6000)(60)(60)}{(6.0)(10^6)}} = 1.893$$

$$C_1 = 3.51 \qquad C_2 = 2.14$$

$$M_{FBC} = 3185 \text{ in.-lb} \qquad M_{FCB} = -3185 \text{ in.-lb}$$

For span CC'

$$\alpha = 1.604 \qquad C_1 = 3.63 \qquad C_2 = 2.08$$

$$M_{FCC'} = 1690 \text{ in.-lb}$$

From symmetry it is known that

$$\theta_{C'} = -\theta_C$$

therefore

$$M_{CC'} = 3.63K\theta_C + 2.08K\theta_{C'} = 1.55K\theta_C$$

If the stiffness factor of span CC' is taken as $1.55K$ the rotation of $\theta_{C'}$ is taken into consideration and no carry-over is necessary.

Distribution and Carry-over Factors

Member	C	K	CK	r	Carry-over
			Joint B		
BA	2.55	$\frac{4.0}{80}$	0.1275	0.353	0
BC	3.51	$\frac{4.0}{60}$	0.2340	0.647	0.61
			$\Sigma CK = 0.3615$	1.000	
			Joint C		
CB	3.51	$\frac{4.0}{60}$	0.2340	0.624	0.61
CC'	1.55	$\frac{4.0}{44}$	0.1410	0.376	
			$\Sigma CK = 0.375$	1.000	

The distribution of the fixed-end moments is shown in Fig. 158. These operations are performed in the usual way.

When the proper coefficients and fixed-end moments are substituted

in equations 141a and b the end moments for the continuous beam in Fig. 158 can be expressed in the following form:

$$M_{BA} = 0.1275\theta_b - 7487$$
$$M_{BC} = 0.234\theta_b + 0.143\theta_c + 3185$$
$$M_{CB} = 0.143\theta_b + 0.234\theta_c - 3185$$
$$M_{CC'} = 0.141\theta_c + 1690$$

When the above expressions for end moments are substituted in the equilibrium equations

$$M_{BA} + M_{BC} = 0$$
$$M_{CB} + M_{CC'} = 0$$

the following equations are obtained

$$0.3620\theta_b + 0.143\theta_c = 4302$$
$$0.143\theta_b + 0.375\theta_c = 1495$$

from which

$$\theta_b = 12{,}130 \qquad \theta_c = -648$$

If these values of θ are substituted back in the slope-deflection equations the numerical values of the end moments become

$$M_{BA} = 5937 \text{ in.-lb} \qquad M_{CB} = -1602 \text{ in.-lb}$$

90. Bending Moments in Beams with Axial Loads. The bending moment at any section of a beam that is subjected to both transverse and axial loads is obtained by substituting the proper value of y from equation 136 into equation 133, or 143c into 143a. The displacement y for a beam with uniform load w and an axial compressive load P, both constant across the span, can be expressed by means of equations 137 and 138 in the form

$$y = \frac{1}{P}\left(A' \cos k\alpha + B' \sin k\alpha - M_s - \frac{wl^2}{\alpha^2}\right) \tag{149}$$

in which

$$A' = M_{ab} + \frac{wl^2}{\alpha^2} \tag{150a}$$

$$B' = -\frac{A' \cos \alpha + \left(M_{ba} - \dfrac{wl^2}{\alpha_2}\right)}{\sin \alpha} \tag{150b}$$

$$\alpha = \sqrt{\frac{Pl^2}{EI}}$$

$$k = \frac{x}{l}$$

The bending moment M_x at a distance x from the left support is, therefore,

$$M_x = M_s + Py = A' \cos k\alpha + B' \sin k\alpha - \frac{wl^2}{\alpha^2} \tag{151}$$

As A' and B' are constants the bending moment can be calculated for any value of $k\alpha$.

The maximum bending moment is obtained when the value of k satisfies the condition

$$\frac{dM_x}{dk} = -\alpha A' \sin k\alpha + \alpha B' \cos k\alpha = 0 \tag{152a}$$

or

$$\tan k\alpha = \frac{B'}{A'} \tag{152b}$$

Example 58. The bending moments at several sections in span BC of the continuous beam in Fig. 158 will be calculated by means of equation 151.

$$\alpha = 1.893 \qquad \frac{wl^2}{\alpha^2} = \frac{(-10)(60)(60)}{(1.893)^2} = -10{,}040$$

$$\cos \alpha = -0.31665 \qquad M_{BC} = 5935$$

$$\sin \alpha = 0.94853 \qquad M_{CB} = -1602$$

$$A' = 5935 - 10{,}040 = -4105$$

$$B' = -\frac{(-4105)(-0.31665) + (-1602 + 10{,}040)}{0.94853} = -10{,}270$$

Therefore,

$$M_x = -4105 \cos k\alpha - 10{,}270 \sin k\alpha + 10{,}040$$

For various values of k, M_x has the following values:

$k = \frac{x}{l}$	$k\alpha$	M_x
0	0	5935 in.-lb
0.2	0.379	2430
0.4	0.7572	10
0.6	1.1358	−1000
0.8	1.5144	−440
1.0	1.893	1600

For the minimum moment,

$$\tan k\alpha = \frac{-10{,}270}{-4105} = 2.50$$

$$k\alpha = 1.19$$

$$k = \frac{1.19}{1.89} = 0.629$$

$$M_{\min} = (-4105)(0.372) - (10{,}270)(0.928) + 10{,}040 = -1010 \text{ in.-lb}$$

91. Beams with Variable Cross Section Subjected to Axial Loads. When both the bending moment M_x and the moment of inertia I in equation 134 vary with the distance x, a direct solution for the equation of the elastic curve is not often feasible. For such problems, indirect methods, that is, solutions by trial, are frequently the most practical approach. The indirect procedure is particularly satisfactory when successive trial values indicate a rapidly converging series. The general procedure for a solution by trial when the axial load is known is straightforward and easy to understand, even though the numerical work is laborious. It consists of estimating the bending moment Py due to the axial load, by assuming some elastic curve for the member, and adding this moment to M_s, the moment for a straight member. If the elastic curve which is then calculated from this $\frac{M}{EI}$ diagram agrees with the assumed values, the solution is correct. If the calculated displacements differ from the assumed values, the solution must be repeated by using a different elastic curve until the two values agree. The amount of work required will depend somewhat upon the ability of the designer to approximate the final values.

For the first approximation, the following procedure will frequently give satisfactory results. Remove the axial load and draw the $\frac{M_s}{EI}$ diagram which is used as the applied load on the conjugate beam. Calculate the bending moments in the conjugate beams at several points which are equal to the displacements y_0 in the actual beam. Now apply the axial load P and assume that the displacements y_0 are increased in proportion to the increase in bending moment, that is,

$$\frac{y}{y_0} = \frac{M_s + Py}{M_s}$$

or

$$y = \frac{y_0}{1 - \dfrac{Py_0}{M_s}}$$

These estimated values of y are then used to draw a new $\frac{M}{EI}$ diagram, from which the first calculated values of y are determined. If these calculated values of y differ too much from the estimated ones, they should be used as a new set of estimated displacements and the operation should be repeated. In most problems, two approximations are usually sufficient.

Many short cuts and time-saving procedures are, of course, possible in solutions by trial. The reader is referred to a paper by N. M. Newmark, "Numerical Procedure for Computing Deflections, Moments, and Buckling Loads," in the *Trans. Am. Soc. C. E.*, Vol. 108, 1943, for some excellent suggestions for arrangement of the calculations and for the use of trial solutions in general. Most designers encounter the problem so seldom that they would probably prefer a procedure with which they are familiar even though a more rapid method is possible.

Example 59. The coefficients C_1, C_2, and C_3 for the general slope-deflection equations will be calculated for the beam shown in Fig. 159

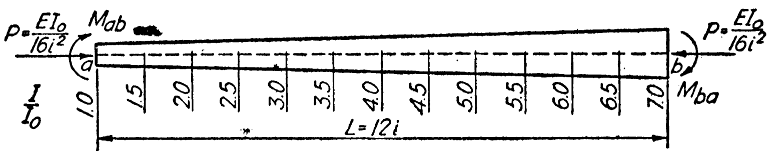


FIG. 159.

by the method of successive trials. This problem is identical with the one solved on p. 1221 of Ref. 9·2. The following steps will be used in the solution:

(*a*) The displacements y_0 are calculated for a moment at the left end, M_{ab}, equal to 12 and with the axial load P removed. For calculating y_0 the $\frac{M_s}{EI}$ diagram is divided into areas as indicated in Fig. 160*a*, an arrangement that makes the calculated displacements slightly larger than the correct value. The correction, which is one-sixth of the product of the concentrated load at the point and the increment i, can be made if desired, but it hardly seems necessary in this problem. The values of y_0 are given in Fig. 160*b*.

(*b*) After the values of y_0 have been calculated, the estimated values of y, the displacements when the axial load P is also acting, are determined from the relation

$$y = \frac{y_0}{1 - \dfrac{Py_0}{M_s}}$$

For example, at point 6

$$y_6 = \frac{\dfrac{35.21i^2}{EI_0}}{1 - \left(\dfrac{EI_0}{16i^2}\right)\left(\dfrac{35.21i^2}{EI_0 6}\right)} = 55.7\frac{i^2}{EI_0} \quad \text{say } 56$$

The revised $\frac{M}{I}$ value at point 6 is therefore

$$\frac{M}{I} = \frac{M_s + Py}{4I_0} = \frac{6.0 + \left(\frac{EI_0}{16i^2}\right)\left(\frac{56i^2}{EI_0}\right)}{4I_0} = \frac{2.38}{I_0}$$

The $\frac{M}{I}$ values at the other points are estimated in the same manner. These values are calculated roughly as they are estimated values only. Estimated values of y are shown in Fig. 160c.

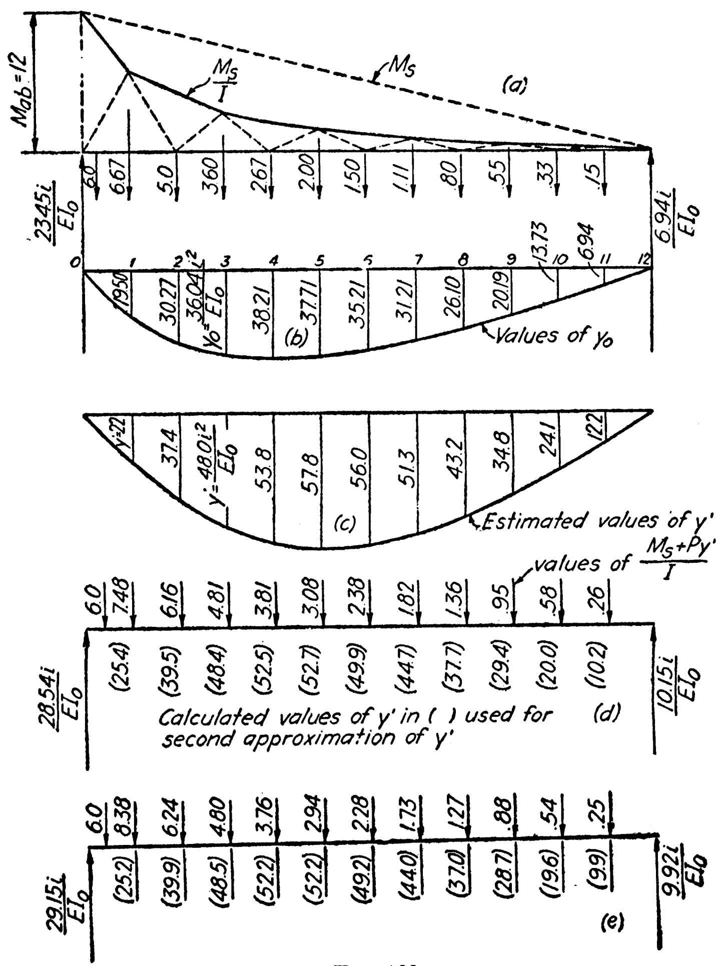


FIG. 160.

(*c*) After the concentrated loads on the conjugate beam have been revised, see Fig. 160*d*, these new loads are used in calculating another set of displacements which are given in parentheses. A comparison of these values with the estimated values in Fig. 160*c* shows that the estimated values were approximately 10 per cent too large.

(*d*) The concentrated loads on the conjugate beam in Fig. 160*e* were determined from the calculated displacements in Fig. 160*d*. The second set of calculated displacements given in parentheses are sufficiently close to the first set of calculated values to make a further revision unnecessary. The end rotations for a unit end moment at the left end are, therefore,

$$\beta_{a1} = \frac{29.15}{144}\frac{L}{EI_0} = 0.2024\frac{L}{EI_0}$$

$$\beta_{b1} = \frac{9.92}{144}\frac{L}{EI_0} = 0.0689\frac{L}{EI_0}$$

(*e*) In Figs. 161*a*, *b*, *c*, *d*, and *e* is shown the corresponding solution for a moment of 12 applied at the right end. The values recorded in these diagrams were obtained in the same manner as was explained in steps (*a*)–(*d*) for Fig. 160. From the final values in Fig. 161*e*, the end rotations for a unit moment acting at the right end are

$$\beta_{a2} = \frac{9.73}{144}\frac{L}{EI_0} = 0.0676\frac{L}{EI_0}$$

$$\beta_{b2} = \frac{11.38}{144}\frac{L}{EI_0} = 0.079\ \frac{L}{EI_0}$$

(*f*) After the end rotations for the unit end moments have been determined, the coefficients can be calculated from equations 58*a*, *b*, and *c* as for beams with variable moments of inertia. From the above β values, we obtain

$$A = (0.2024)(0.079) - (0.0683)^2 = 0.01133$$

$$C_1 = \frac{0.079}{A} = 6.97$$

$$C_2 = \frac{0.0683}{A} = 6.03$$

$$C_3 = \frac{0.2024}{A} = 17.86$$

The stiffness factors are therefore $\frac{6.97EI_0}{L}$ and $\frac{17.86EI_0}{L}$, and the carry-over factors $\frac{6.03}{6.97} = 0.865$ and $\frac{6.03}{17.86} = 0.338$.

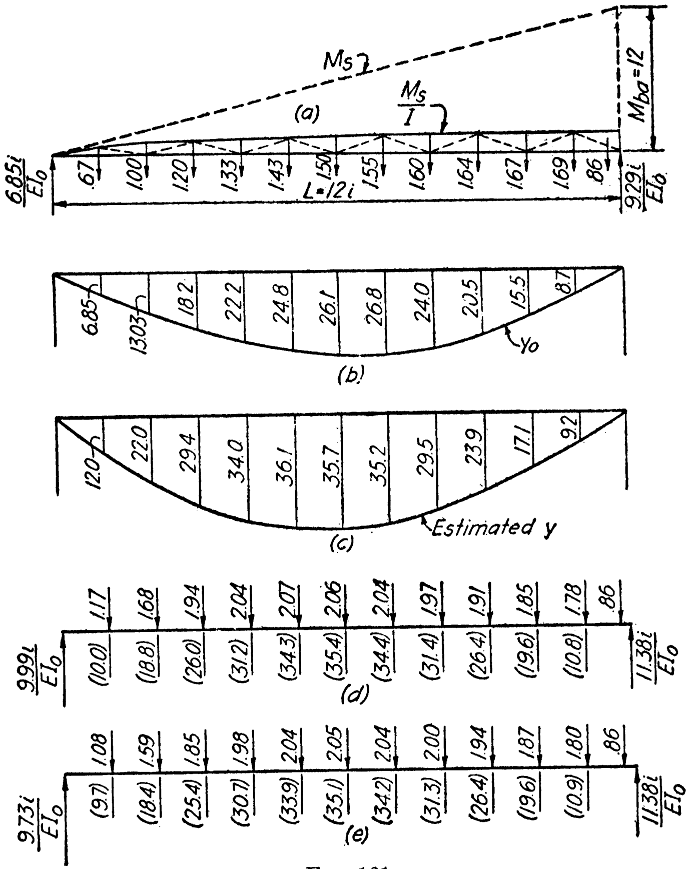


FIG. 161.

92. Suspension Bridges. The analysis and design of suspension bridges frequently involve special problems that require more comprehensive treatment than is given in the following discussion. The fundamental equations for ordinary conditions will be considered here, and the reader who desires to continue his study of the subject further should consult the references listed at the end of the chapter. As the suspension bridge is ordinarily constructed, it consists of the cables, towers, anchorages, stiffening trusses or girders, suspenders, and road-

way. The suspension bridge shown in Fig. 162 has three spans in which the stiffening trusses are hinged at the towers and in which the cables are loaded in all spans. Other arrangements that are sometimes used include continuous stiffening trusses, unloaded cables in the end spans,

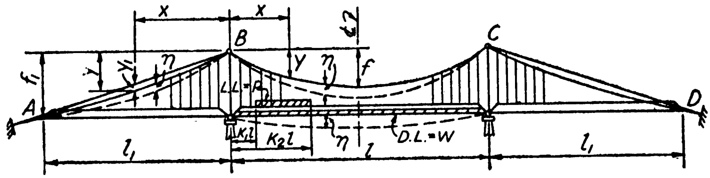


Fig. 162.

self-anchored bridges, and various types of cables. For the conventional type of suspension bridge that is shown in Fig. 162, the following assumptions are commonly made in the analysis:

(*a*) The dead weight is uniformly distributed and is carried entirely by the cable. For this loading condition the shape of the cable is parabolic and the cable must be placed in an assigned position during construction.

(*b*) The deflection η of the stiffening truss for live loads is the same as for the cable. Any change in length of the suspenders is therefore neglected.

(*c*) The proportion of the live load that is taken by the cable, designated in Fig. 163 by q, is often assumed to be uniformly distributed so as to simplify the mathematical solution. This assumption means that the shape of the cable is taken as a parabola for combined dead and live loads. The error involved in this assumption depends upon the ratio of live to dead load as well as the physical characteristics of the cable and stiffening truss. For large structures in which the dead weight is a major portion of the total, the error is small.

In the solution in which the displacements are represented by trigonometric series no assumption as to the distribution of q is necessary. For this reason, as well as others, such a solution has many advantages.

(*d*) The horizontal component H of the cable stress is assumed to be constant for all spans unless the cable is fixed to unusually stiff towers.

(*e*) In calculating the external work that occurs when the live load is applied to the structure, it is assumed that the deflection is proportional to the applied live load. This assumption is permissible for the ratio of live to dead load and for the sag-span ratios ordinarily used.

93. Calculation of the Horizontal Component H. The horizontal component H of the stress in the cable is usually determined first in the analysis of suspension bridges. From assumption (*a*) in Art. 92 the uniform dead load w is taken entirely by the cable and the value of H_D

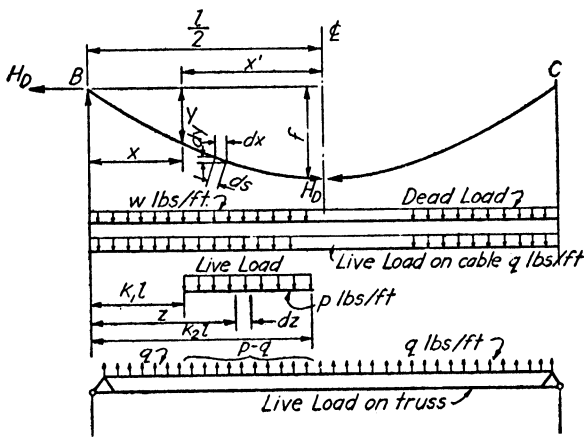


FIG. 163.

can therefore be calculated directly from equilibrium conditions. Taking moments about point B, Fig. 163, we obtain

$$H_D = \frac{wl^2}{8f} \tag{153a}$$

in which f = maximum cable sag in center span.
l = length of center span.
w = uniformly distributed dead load.

The equation of the cable for point B as origin is

$$y = \frac{4fx}{l^2}(l - x) = \frac{wx}{2H_D}(l - x) \tag{153b}$$

The sag-span ratio $\frac{f}{l}$ is known from preliminary studies. In the spinning of the cables the wires must be set at the proper elevation so that the desired sag-span ratio is obtained at normal temperatures after the roadway is completed. Considerable calculations are required to determine the correct elevation of the wires during construction (see Ref. 9·3).

As the live load is carried by both the stiffening truss and the cable, the structure is once statically indeterminate for this loading condition.

The additional horizontal component H_L in the cable, due to the live load, is usually selected as the redundant quantity to be determined. The total horizontal component H is therefore equal to

$$H = H_D + H_L = H_D(1 + \beta) \tag{154}$$

where

$$\beta = \frac{H_L}{H_D}$$

It will be shown that the numerical value of H depends upon the change in temperature as well as upon the magnitude and distribution of the live and dead loads. By equating the external work done on the cable to the corresponding strain energy existing in the cable, an equation is obtained from which the value of the horizontal component H can be determined by trial. The external work, for any portion ds of the cable, that is caused by applying an increment of live load q to the dead load w is

$$dW_e = w\eta\, dx + \frac{q}{2}\eta\, dx = \left(w + \frac{q}{2}\right)\eta\, dx \tag{155a}$$

This equation obviously implies a linear relation between q and the displacement η. Such a linear variation does not theoretically exist, and the only justification for using it is that the displacements that are calculated from this assumption are found to be practically linear for actual structures (see Ref. 9·9). This assumption would not be warranted for the analysis of the horizontal wire in Article 84, for in such a member the displacements are far from being proportional to the load.

The total external work for all spans is

$$W_e = \sum \int_0^l \left(w + \frac{q}{2}\right)\eta\, dx \tag{155b}$$

in which the Σ means that the integration is made for all spans.

The internal work or strain energy in any element ds of the cable is

$$dW_i = \left(T_D + \frac{T_L}{2}\right)\left(\frac{T_L}{AE} + \alpha t\right) ds \tag{156a}$$

in which the tension T in the cable at any section is equal to

$$T = H \sec\theta = H\frac{ds}{dx} \tag{156b}$$

αt = deformation per unit length due to change in temperature.

Therefore,

$$dW_i = \left(H_D + \frac{H_L}{2}\right)\frac{ds}{dx}\left(\frac{H_L}{AE}\frac{ds}{dx} + \alpha t\right) ds$$

or

$$dW_i = H_D\left(1 + \frac{\beta}{2}\right)\left[\frac{H_D}{AE}\beta\left(\frac{ds}{dx}\right)^2 + \alpha t\frac{ds}{dx}\right] ds \qquad (156c)$$

The total internal work for all spans is

$$W_i = H_D\left(1 + \frac{\beta}{2}\right)\sum\int_0^l \left[\frac{H_D}{AE}\beta\left(\frac{ds}{dx}\right)^2 + \alpha t\frac{ds}{dx}\right] ds \qquad (157)$$

in which the integration is made for all spans.

For equilibrium of any portion of the cable,

$$H_D\frac{d^2y}{dx^2} = -w \qquad (158a)$$

and

$$(H_D + H_L)\frac{d^2(y + \eta)}{dx^2} = -(w + q) \qquad (158b)$$

from which

$$H_D\frac{d^2y}{dx^2} + H_D\frac{d^2\eta}{dx^2} + H_L\frac{d^2y}{dx^2} + H_L\frac{d^2\eta}{dx^2} = -w - q \qquad (158c)$$

Substituting the value of w in equation $158a$ into equation $158c$ gives

$$q = -\frac{H_L}{H_D}H_D\frac{d^2y}{dx^2} - H_D(1 + \beta)\frac{d^2\eta}{dx^2}$$

or

$$q = \beta w - H_D(1 + \beta)\frac{d^2\eta}{dx^2} \qquad (158d)$$

After combining equations $155b$, 157, and $158d$, we obtain

$$H_D\left(1 + \frac{\beta}{2}\right)\sum\int_0^l \left[\frac{H_D}{AE}\beta\left(\frac{ds}{dx}\right)^2 + \alpha t\frac{ds}{dx}\right] ds$$

$$= \sum\int_0^l \left[w + \frac{w\beta}{2} - \frac{H_D(1 + \beta)}{2}\frac{d^2\eta}{dx^2}\right]\eta\, dx \qquad (159)$$

from which the value of β can be calculated by trial after the displacement η has also been expressed in terms of β.

In solving equation 159 it is necessary to evaluate the integrals

$$\int\left(\frac{ds}{dx}\right)^2 ds \qquad \text{and} \qquad \int\left(\frac{ds}{dx}\right) ds$$

for all spans. The solution of these integrals for a parabolic curve with origin at the center of the span gives

For center span

$$y = \frac{4fx'^2}{l^2} \qquad \frac{dy}{dx'} = \frac{8fx'}{l^2} \qquad ds = dx'\sqrt{1 + \frac{64f^2x'^2}{l^4}}$$

$$\int_0^l \left(\frac{ds}{dx'}\right)^2 ds = 2\int_0^{l/2}\left(1 + \frac{64f^2x'^2}{l^4}\right)^{3/2} dx'$$

$$= l\left\{\frac{1}{4}\left(\frac{5}{2} + \frac{16f^2}{l^2}\right)\left(1 + \frac{16f^2}{l^2}\right)^{1/2} + \frac{3}{32}\frac{l}{f}\log_e\left[\frac{4f}{l} + \left(1 + \frac{16f^2}{l^2}\right)^{1/2}\right]\right\} \qquad (160a)$$

$$\int_0^l \frac{ds^2}{dx'} = 2\int_0^{l/2}\left(1 + \frac{64f^2x'^2}{l^4}\right) dx' = l\left[1 + \frac{16}{3}\frac{f^2}{l^2}\right] \qquad (160b)$$

For side spans (see Fig. 164)

$$\int_0^l \left(\frac{ds}{dx'}\right)^2 ds = \int_{x'_a}^{x'_b}\left[1 + \frac{64f'^2}{l'^4}x'^2\right]^{3/2} dx' \qquad (160c)$$

$$\int_0^l \left(\frac{ds}{dx'}\right) ds = \int_{x'_a}^{x'_b}\left[1 + \frac{64f'^2}{l'^4}x'^2\right] dx' \qquad (160d)$$

Approximate values for the above integrals can be easily obtained by expanding the expressions by the binomial theorem and then integrating the various terms. This operation will be left to the reader.

94. Value of Displacement η for Center Span. Before equation 159 can be solved for β, the terms η and $\frac{d^2\eta}{dx^2}$ must be expressed in terms of β, the dead load w, the live load p, and the properties of the stiffening truss. This relation is obtained from the fundamental equation of the elastic curve of the stiffening truss in practically the same manner as for the beams in Article 86. From Fig. 163, the bending moment due to the live load p at any section through the stiffening truss at a distance x from the left end is

$$M = M_s - (H_D + H_L)\eta - H_Ly \qquad (161a)$$

and, therefore,

$$\frac{d^2\eta}{dx^2} = -\frac{M}{EI} = \frac{-M_s + (H_D + H_L)\eta + H_Ly}{EI} \qquad (161b)$$

If we let

$$a^2 = \frac{H_D + H_L}{EI} = \frac{H_D(1 + \beta)}{EI} \qquad (161c)$$

then

$$\frac{d^2\eta}{dx^2} - a^2\eta = -\frac{M_s}{EI} + \frac{H_Ly}{EI} \qquad (161d)$$

The solution of this equation, as given in Art. 86, is

$$\eta = A \cosh ax + B \sinh ax + \frac{1}{EI}\left(\frac{M_s}{a^2} + \frac{1}{a^4}\frac{d^2M_s}{dx^2} + \cdots\right)$$
$$- \frac{H_L}{EI}\left(\frac{y}{a^2} + \frac{1}{a^4}\frac{d^2y}{dx^2} + \cdots\right) \quad (162)$$

The integration constants A and B must satisfy the boundary conditions and the conditions of continuity of the elastic curve for any particular loading. For a partial uniform loading on the roadway, the bending moment M_s and therefore the displacement η must be expressed by three different equations, and, consequently, there are six integration constants to evaluate. This requirement makes the exact solution of the differential equation extremely laborious. The only alternative, as has already been seen, is to express the displacement η by a trigonometric series, as was done by S. Timoshenko and other investigators. (See Ref. 9·6, 9·8, and 9·11.)

As in Article 88, the displacement η is expressed by the series

$$\eta = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{2\pi x}{l} + \cdots = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} \quad (163a)$$

$$\frac{d^2\eta}{dx^2} = -\frac{\pi^2}{l^2}\sum_{n=1}^{\infty} n^2 a_n \sin\frac{n\pi x}{l} \quad (163b)$$

In Article 88 it was shown that the bending moment in a simply supported beam for a concentrated load W at a distance c from the left support is expressed by the Fourier series,

$$M_s = \frac{2Wl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi c}{l}\sin\frac{n\pi x}{l} \quad (164)$$

A uniform load that extends from $k_1 l$ to $k_2 l$ (Fig. 163) is equivalent to a number of concentrated loads $p\,dz$ which are at a distance $z = c$ from the left support. Therefore, the moment caused by the uniform load is

$$M_s = \frac{2pl}{\pi^2}\int_{z=k_1 l}^{z=k_2 l}\sum\frac{1}{n^2}\frac{\sin n\pi z}{l}\,dz \sin\frac{n\pi x}{l} \quad (165a)$$

which, after integration, gives

$$M_s = \frac{2pl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(\cos n\pi k_1 - \cos n\pi k_2)}{n^3}\sin\frac{n\pi x}{l} \quad (165b)$$

The equation for the original position of the cable

$$y = \frac{4f}{l^2}(lx - x^2) = \frac{w}{2H_D}(lx - x^2) \tag{166a}$$

is readily developed into the Fourier series

$$y = \frac{2wl^2}{H_D\pi^3}\sum_{n=1}^{\infty}\frac{(1 - \cos n\pi)}{n^3}\sin\frac{n\pi x}{l} \tag{166b}$$

When the above Fourier series are substituted in equation 161d we obtain

$$-\frac{\pi^2}{l^2}\sum_{n=1}^{\infty} n^2 a_n \sin\frac{n\pi x}{l} - \frac{H_D(1+\beta)}{EI}\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$$

$$= -\frac{2pl^2}{\pi^3 EI}\sum_{n=1}^{\infty}\frac{(\cos n\pi k_1 - \cos n\pi k_2)}{n^3}\sin\frac{n\pi x}{l}$$

$$+\frac{2H_L wl^2}{EIH_D\pi^3}\sum_{n=1}^{\infty}\frac{(1-\cos n\pi)}{n^3}\sin\frac{n\pi x}{l} \tag{167a}$$

For any value of $\sin\frac{n\pi x}{l}$, equation 167a requires that the coefficients a_n have the value

$$-a_n\left[\frac{n^2\pi^2}{l^2} + \frac{H_D(1+\beta)}{EI}\right] = -\frac{2pl^2}{\pi^3 EI}\left(\frac{\cos n\pi k_1 - \cos n\pi k_2}{n^3}\right)$$

$$+\frac{2H_L wl^2}{EIH_D\pi^3}\left(\frac{1-\cos n\pi}{n^3}\right) \tag{167b}$$

or since $\frac{H_L}{H_D} = \beta$

$$a_n = \frac{2l^4[p(\cos n\pi k_1 - \cos n\pi k_2) + \beta w(1 - \cos n\pi)]}{n^3\pi^3[n^2\pi^2 EI + H_D(1+\beta)l^2]} \tag{167c}$$

The above expression for the coefficients a_n contains the term β which is also the unknown quantity in equation 159. However, as equation 159 must be solved by trial the calculation of the coefficients a_n for any assumed value of β does not add much work to the entire solution. Once equation 159 has been solved for β, the cable stress, deflections, bending moments, and shear in the stiffening truss are readily ascertained.

After substituting the value of η from equation 163a into equation 159, the term

$$\int_0^l \frac{H_D(1+\beta)}{2}\,\eta\,\frac{d^2\eta}{dx^2}\,dx \tag{168a}$$

becomes for any value of a_n

$$\frac{H_D(1+\beta)}{2}\int_0^l -\frac{n^2\pi^2}{l^2}\,a_n^2\sin^2\frac{n\pi x}{l}\,dx$$

which is equal to

$$\frac{H_D(1+\beta)}{2}\left[-\left(\frac{n^2\pi^2}{l^2}\,a_n^2\right)\left(\frac{l}{2}\right)\right] \tag{168b}$$

Consequently the value of the term for all values of a_n is

$$-\frac{H_D(1+\beta)\pi^2}{4l}\sum n^2a_n^2 \tag{168c}$$

95. Value of the Displacement η for the Side Spans. For the side spans the equilibrium of the cable requires that

$$H_D\frac{d^2y}{dx^2} = -w_1 \tag{169a}$$

which after integrating twice gives

$$H_Dy = -\frac{w_1x^2}{2} + C_1x + C_2 \tag{169b}$$

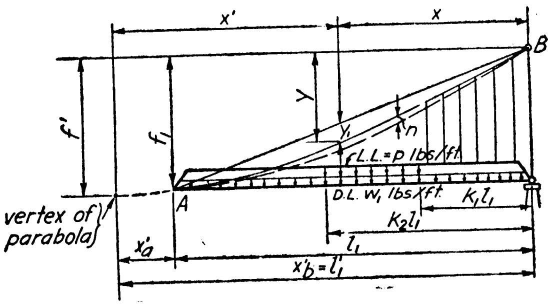


FIG. 164.

If point B (Fig. 164) at the top of the tower is selected as the origin,

$$y = 0 \quad \text{when } x = 0 \qquad \text{or} \quad C_2 = 0$$

$$y = f_1 \quad \text{when } x = l_1 \qquad \text{or} \quad C_1 = H_D\frac{f_1}{l_1} + \frac{w_1l_1}{2}$$

Therefore

$$H_D\left(y - \frac{f_1}{l_1}x\right) = -\frac{w_1x^2}{2} + \frac{w_1l_1}{2}x \tag{169c}$$

but

$$y - \frac{f_1}{l_1}x = y_1$$

Consequently, the equation for the shape of the cable is

$$y_1 = \frac{w_1}{2H_D}(l_1x - x^2) \tag{169d}$$

This equation is identical in form with equation 166a, and therefore, if y is replaced by y_1, l by l_1, I by I_1, the equation for the elastic curve of the stiffening truss in a side span can be obtained in the same manner as for the center span. That is, if for the side span

$$\eta = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l_1} \tag{170a}$$

then

$$b_n = \frac{2l_1^4[p(\cos n\pi k_1 - \cos n\pi k_2) + \beta w_1(1 - \cos n\pi)]}{n^3\pi^3[n^2\pi^2EI_1 + H_D(1 + \beta)l_1^2]} \tag{170b}$$

Also, as for the center span, the term

$$\int_0^{l_1} \frac{H_D(1 + \beta)}{2}\eta\frac{d^2\eta}{dx^2}dx = -\frac{H_D(1 + \beta)\pi^2}{4l_1}\sum_{n=1}^{\infty} n^2b_n^2 \tag{171}$$

96. Flexible Arches. In Chapter VIII the analysis of arches and curved members was made by the elastic theory; that is, the displacement of the arch axis was neglected. However, in the design of long-span arch bridges the stresses that are calculated by the elastic theory are found to be considerably less than their true values as given by the deflection theory, which considers the movement of the arch axis. An exact mathematical solution of an arch rib by the deflection theory is similar to the analysis of a suspension bridge, although the numerical calculations are even more laborious and difficult. The increased difficulty arises in the expression for the internal work in the arch rib, which involves both flexural and direct stresses, whereas the internal work in the cable of a suspension bridge is due only to axial tension. To mitigate these mathematical difficulties many investigators have confined their analysis to special problems in which the equations can

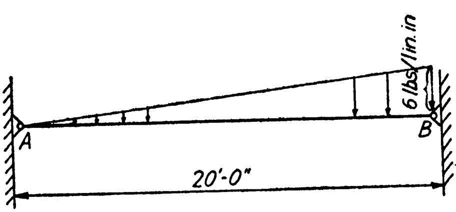
be simplified. In spite of such assumptions the numerical work involves tremendous effort. The references at the end of the chapter will be found useful for further study of these problems.

The longest solid-rib arch bridge that has yet been built is the Rainbow Arch bridge over the Niagara River (Ref. 9·17). It is interesting to note that the maximum positive bending moment at the quarter point of a two-hinged arch rib that was first investigated for this bridge was 63 per cent more for the deflection theory than for the elastic theory. For the fixed arch rib that was actually constructed this moment was about 18 per cent larger by the deflection theory. It is apparent that for structures of this magnitude the deflection of the arch rib must be considered.

In the analysis of the Rainbow Arch bridge, a method of successive approximations was found to be most advantageous. In most solutions by this method, the values of the reactions, shears, and bending moments as determined by the elastic theory are used as the first approximation, and from them a trial position of the arch is determined by either algebraic or graphical methods. The coordinates of the various elements into which the arch axis is divided are then corrected and the numerical operations repeated. The corrections become less for each repetition until the difference can be neglected. This work, which is similar to the procedure in Example 59, can usually be reduced by anticipating some of the corrections in advance. The practical estimation of such corrections is described by the designers of the Rainbow Arch bridge in Ref. 9·17. Every design of bridges of this type will require special treatment.

PROBLEMS

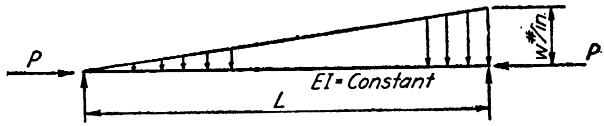
76. A wire whose cross-sectional area is 0.06 in.^{2} is subjected to the loading shown. Determine the total tension in the wire and the maximum sag, if E equals 27×10^6 lb per in.^{2} Assume no initial tension.



PROBLEM 76.

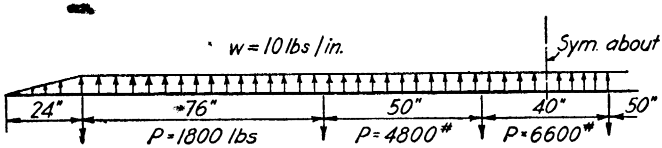
77. Determine the tension in the wire of Problem 76 if an initial tension T_0 of 1000 lb is acting before the radial pressure is applied.

78. Calculate the fixed-end moments for the beam shown by use of trigonometric series as in Example 56. Use α equal to 2.



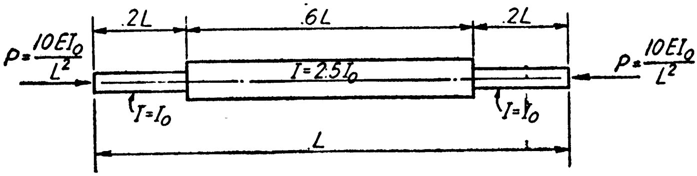
PROBLEM 78.

79. Determine the end moments in the continuous beam for the transverse and axial loads given on the diagram. Draw the bending-moment diagram for span bc. $EI = 8 \times 10^6$ for all spans.



PROBLEM 79.

80. Calculate the coefficients of the slope-deflection equations for the member shown. Use the method of successive approximations as in Example 59.



PROBLEM 80.

81. Determine the fixed-end moments for the beam in Problem 80, for a concentrated load of 1000 lb at the center of the span.

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CHAPTER X

SPECIAL PROBLEMS IN STATICALLY INDETERMINATE STRESSES

Frames with Semi-Rigid Connections

97. Introduction. In many types of structures the beams are riveted or welded to columns and girders by means of angle and tee sections. A typical example of a beam-to-column connection is illustrated in Fig. 165 in which an I-beam frames into an H-column. The flanges of the beam are connected to the flanges of the column by two tee sections while the web is connected by angles. Angles are also commonly used to connect the flanges. The tee and angle sections are fastened to the beam and column by means of rivets, or bolted or welded connections are sometimes used.

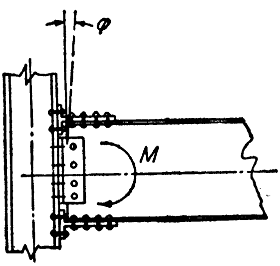

FIG. 165.

Such structural details necessitate a discontinuity in the cross section of the beam that will increase noticeably the strain at the ends of the member. The magnitude of this strain in the connection details cannot be calculated accurately by rational methods as local stress concentration and deformation make the problem too complicated. As the reactive forces between the column and the beam must pass through the rivets, severe local deformation may occur in addition to the distortion of the tee and angle sections. In experimental work it is difficult to evaluate the separate effects of such individual factors as change in length of rivets, local bending of angles or tees, shearing deformation, and local bending of the column flanges. For this reason experimental results for a particular arrangement of connection details can seldom be applied to other types of connections, and consequently it is desirable to have experimental data for each type of connection that is used.

98. Experimental Results. Considerable experimental work has been performed in England and in the United States on the deformation of typical riveted and welded beam connections. The effect of this deformation upon the stresses in beams and columns has also been

studied. Particular mention will be made at this time of the results of a comprehensive research program that are recorded in the *First, Second, and Final Reports of the Steel Structures Research Committee of Great Britain* and to the tests conducted at Lehigh University for the American Institute of Steel Construction. In this research work many tests

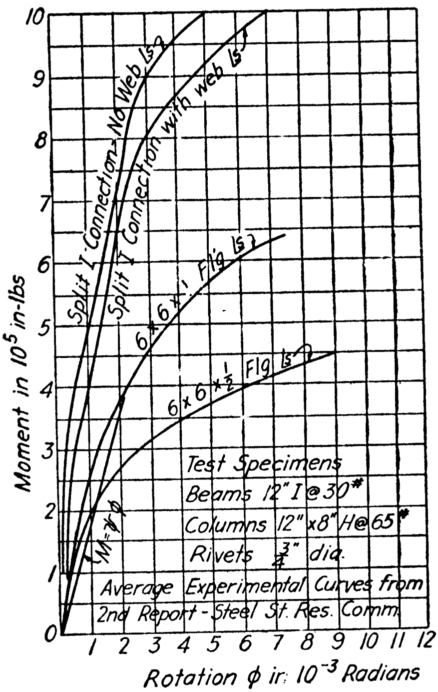

FIG. 166.

were made of the amount of deformation in beam connections similar to the one shown in Fig. 165. The deformation within the connection is measured by the relation between the moment applied to the connection and the rotation ϕ of the end of the beam with respect to the axis of the column. Typical curves recorded in the *Second Report of the Steel Structures Research Committee* are shown in Fig. 166. It is apparent from these curves that no linear relationship exists between the applied moment M and the end rotation ϕ of the beam connection. However, a somewhat rough approximation of the curve over a limited range can be made by a straight line, which, if the slope is selected somewhat low, will give a conservative design.

When the moment-rotation curve for a typical beam connection is

approximated by one or several straight lines, the moment can be expressed by equations of the type

$$M = \psi\phi \tag{172}$$

where ψ is the slope of the straight line. The quantity $\dfrac{M}{\psi}$ is equivalent to $\dfrac{M}{EI}$ in a beam and can be treated as such in the calculations.

99. Slope-Deflection Equations for Semi-Rigid Connections. Once the stiffness factor ψ of the connection is selected from tests it can be used in determining the coefficients for the slope-deflection equations. In these calculations it is convenient to consider the quantity $\dfrac{M}{\psi}$ as a concentrated load on the conjugate beam, which makes the connection equivalent to a change in cross section of the beam. The derivation can then proceed in the manner explained in Article 20, Chapter III, for beams of constant cross section, and Article 46, Chapter VI, for variable cross section, by superimposing the rotations due to the two end moments. A unit moment at end a in member ab, Fig. 167a,

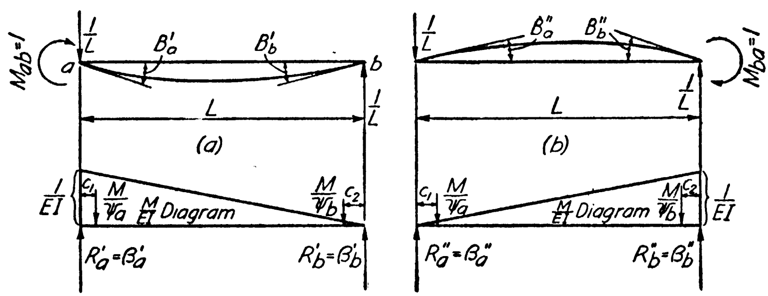

Fig. 167.

would give a load on the conjugate beam as shown, where ψ_a and ψ_b are the stiffness factors for the connections at a and b, respectively. The distances c_1 and c_2 are commonly assumed equal to zero which is equivalent to placing the connection at the center of the joint.

For a unit moment applied at a with the distances c_1 and c_2 equal to zero the end rotations are

$$\beta'_a = \frac{L}{3EI} + \frac{1}{\psi_a} \tag{173a}$$

$$\beta'_b = -\frac{L}{6EI} \tag{173b}$$

When a unit end moment is applied at b, Fig. 167b, the end rotations are

$$\beta_a'' = -\frac{L}{6EI} \tag{174a}$$

$$\beta_b'' = \frac{L}{3EI} + \frac{1}{\psi_b} \tag{174b}$$

The total rotations for any end moments are

$$\theta_a = M_{ab}\left(\frac{L}{3EI} + \frac{1}{\psi_a}\right) - \frac{M_{ba}L}{6EI} \tag{175a}$$

$$\theta_b = -\frac{M_{ab}L}{6EI} + M_{ba}\left(\frac{L}{3EI} + \frac{1}{\psi_b}\right) \tag{175b}$$

When equations 175a and b are solved for M_{ab} and M_{ba}, the usual form of the slope-deflection equations is obtained, that is,

$$M_{ab} = \frac{EI}{L}[C_1\theta_a + C_2\theta_b] \tag{176a}$$

$$M_{ba} = \frac{EI}{L}[C_2\theta_a + C_3\theta_b] \tag{176b}$$

in which

$$C_1 = \frac{12C''}{4C'C'' - 1} \tag{177a}$$

$$C_2 = \frac{6}{4C'C'' - 1} \tag{177b}$$

$$C_3 = \frac{12C'}{4C'C'' - 1} \tag{177c}$$

$$C' = 1 + \frac{3K}{\psi_a} \tag{177d}$$

$$C'' = 1 + \frac{3K}{\psi_b} \tag{177e}$$

$$K = \frac{EI}{L}$$

100. Fixed-End Moments for Semi-Rigid Connections. The fixed-end moments for beams with constant cross section and with semi-rigid connections can be expressed in terms of the fixed-end moments for rigid

connections and the coefficients C_1, C_2, C_3. Let M_{Fab} and M_{Fba} denote the fixed-end moments for rigid connections and M'_{Fab} and M'_{Fba} the corresponding values for semi-rigid connections. These connections are assumed to be at the center of the joints. The values of M'_{Fab} and M'_{Fba} can be obtained from the following equations:

$$M'_{Fab} = \tfrac{1}{6}[M_{Fab}(2C_1 - C_2) + M_{Fba}(2C_2 - C_1)] \qquad (178a)$$

$$M'_{Fba} = \tfrac{1}{6}[M_{Fab}(2C_2 - C_3) + M_{Fba}(2C_3 - C_2)] \qquad (178b)$$

where C_1, C_2, and C_3 have the values given by Equations 177a, b, and c.

For a uniformly distributed load w across the entire span, the values of the fixed-end moments are

$$M'_{Fab} = -\frac{wL^2}{24}(C_1 - C_2) \qquad (179a)$$

$$M'_{Fba} = \frac{wL^2}{24}(C_3 - C_2) \qquad (179b)$$

Once the values of the ψ terms are selected the analysis of any structural frame is made in the same manner as for members with variable moments of inertia.

Example 60. The frame in Fig. 168 will be analyzed for semi-rigid connections between the beams and columns and for a condition of half

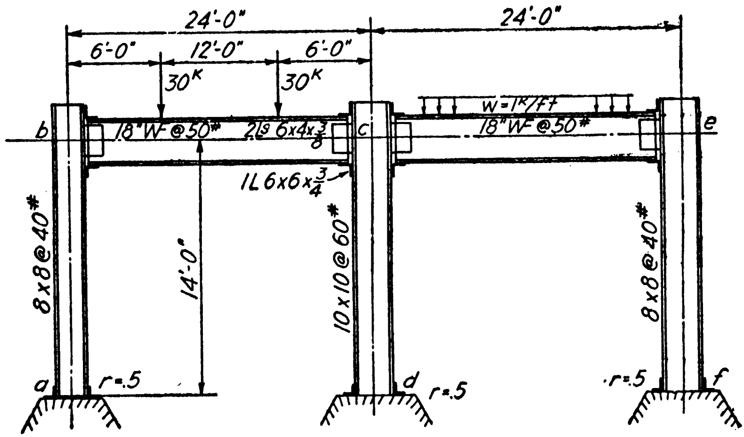

FIG. 168.

fixity at the base of the columns. From a study of test results the value of ψ for all connections will be selected as

$$\psi = 100 \times 10^6 \text{ in.-lb}$$

Therefore we obtain for the 18-in. WF @ 50-lb beams from equations 177*d* and *e*

$$C' = C'' = 1 + \frac{(3)(30)(10^6)(800)}{(24)(12)(100)(10^6)} = 3.50$$

From equations 177*a*, *b*, and *c*

$$C_1 = C_3 = \frac{(12)(3.50)}{(4)(3.50)^2 - 1} = 0.875$$

$$C_2 = \frac{6}{48} = 0.125$$

For span *bc*

$$M_{Fbc} = -\frac{(30)(6)(18)^2}{(24)^2} - \frac{(30)(18)(6)^2}{(24)^2} = -135 \text{ ft-kips}$$

$$M_{Fcb} = +135 \text{ ft-kips}$$

From equations 178*a* and *b*

$$M'_{Fbc} = \tfrac{1}{6}[-135(1.75 - 0.125) + 135.0(0.25 - 0.875)] = -50.63 \text{ ft-kips}$$

$$M'_{Fcb} = +50.63 \text{ ft-kips}$$

From equations 179*a* and *b*

$$M'_{Fce} = -\frac{(1.0)(24)(24)}{(24)}(0.875 - 0.125) = -18.0 \text{ ft-kips}$$

$$M'_{Fec} = +18.0 \text{ ft-kips}$$

Distribution and Carry-over Factors

Joint b

Member	C	$K = \frac{I}{L}$	CK	r	Carry-over
ba	3.5	0.871	3.05	0.556	$\frac{2(1-0.5)}{4-0.5} = 0.286$
bc	0.875	2.78	2.44	0.444	$\frac{0.125}{0.875} = 0.143$
		$\Sigma CK =$	5.49	1.000	

Joint c

Member	C	K	CK	r	Carry-over
cb	0.875	2.78	2.44	0.203	0.143
ce	0.875	2.78	2.44	0.203	0.143
cd	3.5	2.04	7.15	0.594	0.286
		$\Sigma CK =$	12.03	1.00	

The distribution of the fixed-end moments is shown in Fig. 169a. The numerical calculations are the same as for beams with rigid connections. From the end moments in the columns, the horizontal restraining force at b is found to be

$$F = 1.90 + 0.87 - 2.64 = 0.13 \text{ kips}$$

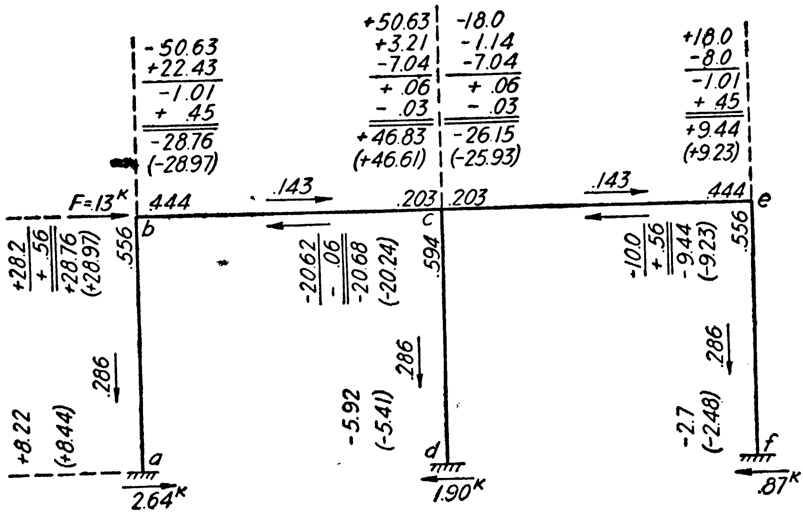

FIG. 169a.

This restraining force is removed in the usual manner by assuming that points b, c, and e have some horizontal displacement, say

$$E\Delta = 100 \text{ units}$$

Therefore

$$M_{Fab} = M_{Fba} = M_{Fef} = M_{Ffe} = \frac{(6)(0.871)(100)}{14} = 37.4$$

$$M_{Fcd} = M_{Fdc} = \frac{(6)(2.04)(100)}{14} = 87.3$$

The distribution of these fixed-end moments is shown in Fig. 169b. It should be noticed that a condition of half fixity is maintained at the base of all columns by first distributing the fixed-end moment at the base, then using a coefficient of 3.5 in calculating the distribution factor at the top of the column and finally a carry-over factor of 0.286 back to the base. The force F' necessary to give the final end moments is

$$F' = 1.995 + 4.41 + 1.995 = 8.40 \text{ kips}$$

Therefore, the correct moments are obtained by adding, to the moments in Fig. 169*a*, $\frac{0.13}{8.40}$ times the moments in Fig. 169*b*.

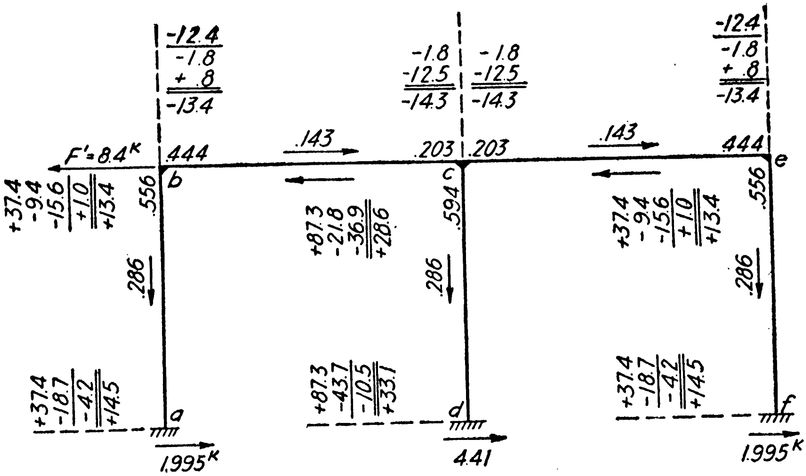

FIG. 169*b*.

101. Discussion of Semi-Rigid Connections. No attempt has been made in the preceding articles to describe a general procedure for the design of beams with semi-rigid connections. Most designers have the idea that the average beam connection becomes entirely flexible under increasing load. This condition cannot be realized because even if the connection reaches a state in which it has no further resistance to rotation it will still maintain a considerable proportion of the end moment that has been developed. In other words, the resistance that has been developed does not disappear after the stiffness of the connection which was designated by ψ approaches zero. However, the amount of resistance developed under different load conditions is uncertain because of the variable nature of the stiffness factor ψ, and such uncertainty has, in the past, encouraged the ignoring of the resistance entirely. There is no doubt that some economy of material can be obtained in certain building frames by adopting minimum values for the stiffness of the connection and then by designing the beams for the end moments as computed from these minimum values. Whether any over-all economy is obtained will depend upon the extent to which the analysis can be shortened by reasonable approximations. It seems likely that such design procedures will be available in the near future.

Calculation of Stresses in Space Frames

102. Deformation Equations for Axial Stress. In Chapter III, equations were derived that expressed the value of the end moments acting on any member in terms of the rotation and translation of the ends of the members. These deformation equations, or slope-deflection equations as they were called, were found to be extremely useful in the solution of statically indetermine frame structures. Similar equations will now be developed for the axial stress in any member in a space frame in terms of the linear motion of the ends of the members.

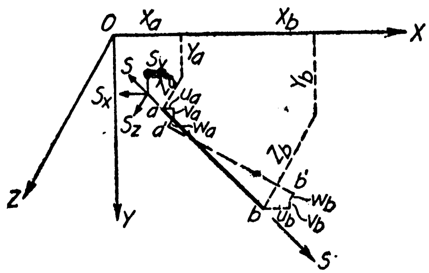

FIG. 170.

Let the member ab, Fig. 170, represent any member in a space structure whose ends a and b have coordinates x_a, y_a, z_a, and x_b, y_b, z_b with respect to some origin O. The linear movement of the end points a and b in the x, y, z directions will be designated by u_a, v_a, w_a and u_b, v_b, w_b, respectively. The projections of the length ab on the three coordinate axes will be called X_{ab}, Y_{ab}, and Z_{ab}, so that

$$X_{ab} = x_a - x_b$$

$$Y_{ab} = y_a - y_b$$

$$Z_{ab} = z_a - z_b$$

In terms of orthogonal projections, the length of the member L is defined by the relation

$$L^2 = X^2 + Y^2 + Z^2 \tag{180a}$$

and, by the usual rules of variation, we obtain the relation

$$2L(\Delta L) = 2X(\Delta X) + 2Y(\Delta Y) + 2Z(\Delta Z)$$

Therefore

$$\Delta L = \frac{1}{L}[(u_a - u_b)X + (v_a - v_b)Y + (w_a - w_b)Z] \tag{180b}$$

since

$$\Delta X = u_a - u_b$$

$$\Delta Y = v_a - v_b$$

$$\Delta Z = w_a - w_b$$

But, for any axial stress S in the member,

$$\Delta L = \frac{SL}{AE}$$

in which AE equals the cross-sectional area times the modulus of elasticity. Therefore

$$S = \frac{AE}{L}\Delta L = \frac{AE}{L^2}[(u_a - u_b)X + (v_a - v_b)Y + (w_a - w_b)Z] \quad (181)$$

The components of the stress S in the direction of the selected axes will therefore be equal to

$$S_x = \frac{SX}{L} = \frac{AE}{L^3}[(u_a - u_b)X^2 + (v_a - v_b)XY + (w_a - w_b)XZ] \quad (182a)$$

$$S_y = \frac{SY}{L} = \frac{AE}{L^3}[(u_a - u_b)XY + (v_a - v_b)Y^2 + (w_a - w_b)YZ] \quad (182b)$$

$$S_z = \frac{SZ}{L} = \frac{AE}{L^3}[(u_a - u_b)XZ + (v_a - v_b)YZ + (w_a - w_b)Z^2] \quad (182c)$$

If

$$Q = \frac{AE}{L^3}$$

then the components of stress for any member can be expressed in terms of the linear displacements of the joints by means of equations 182a, b, and c as soon as the quantities QX^2, QY^2, QZ^2, QXY, QXZ, and QYZ are determined.

103. Equilibrium and Compatibility Equations. As the conditions of statical equilibrium at any joint require that there be no unbalanced components at that point, the following equilibrium equations must be satisfied at each joint:

$$\Sigma S_x - \Sigma F_x = 0 \quad (183a)$$

$$\Sigma S_y - \Sigma F_y = 0 \quad (183b)$$

$$\Sigma S_z - \Sigma F_z = 0 \quad (183c)$$

In the above equations, F_x, F_y, and F_z represent the components of the external loads applied at the joint.

If the values of S_x, S_y, S_z from equations 182 are substituted in the equilibrium equations 183 the following compatibility equations for any joint a are obtained:

$$u_a\Sigma QX^2 + v_a\Sigma QXY + w_a\Sigma QXZ - \Sigma u_n(QX^2)_n - \Sigma v_n(QXY)_n - \Sigma w_n(QXZ)_n - \Sigma F_{xa} = 0 \quad (184a)$$

$$u_a\Sigma QXY + v_a\Sigma QY^2 + w_a\Sigma QYZ - \Sigma u_n(QXY)_n - \Sigma v_n(QY^2)_n - \Sigma w_n(QYZ)_n - \Sigma F_{ya} = 0 \quad (184b)$$

$$u_a\Sigma QXZ + v_a\Sigma QYZ + w_a\Sigma QZ^2 - \Sigma u_n(QXZ)_n - \Sigma v_n(QYZ)_n - \Sigma w_n(QZ^2)_n - \Sigma F_{za} = 0 \quad (184c)$$

In equations 184 the subscript n refers to any joint connected to the joint a by a member. Until the reader is familiar with the procedure it will probably be better for him to substitute the expressions for S_x, S_y, and S_z from equations 182 directly into equations 183. Once the notation is thoroughly understood, however, equations 184 save considerable time.

104. Use of the Deformation Equations. Calculating the stresses in space frames by means of equations 182 and 183 is similar to solving for end moments by the slope-deflection method. The final equations 184 have the linear displacements of the joints as unknowns, and, as there are three such equations for each joint, a unique solution is possible. After the displacements u, v, and w of each joint have been determined, the stress in each member is known from equation 181. This method of analysis requires the solving of $3n$ simultaneous equations, where n is the number of joints free to move in any direction. The solution of this number of equations can become laborious for a structure with many joints. The use of group or block displacements as suggested by Southwell (Ref. 10·6) to simplify the mathematical procedure will be discussed later.

Considerable care must be exercised to maintain a consistent sign convention throughout the calculations. First, the coordinates x, y, and z of each joint must be established correctly with respect to the reference axes. Then the projections X, Y, and Z for each member must be made consistent with the coordinates of the joints by subtracting the coordinates for the opposite end of each member from the coordinates of the joint for which the equations are to be written. That is, if a member ab has coordinates (6, −4, −8) for point a and (−4, 6, −20) for point b, the values of X, Y, and Z to be used in expressing the components of stress at point a are

$$X_{ab} = x_a - x_b = 6 - (-4) = 10$$

$$Y_{ab} = y_a - y_b = (-4) - (6) = -10$$

$$Z_{ab} = z_a - z_b = -8 - (-20) = +12$$

For writing the necessary equations for the stress components at point b, the values are

$$X_{ba} = -4 - (6) = -10$$
$$Y_{ba} = 6 - (-4) = 10$$
$$Z_{ba} = -20 - (-8) = -12$$

That is, the signs for point b are opposite to those for point a.

If the proper signs for X, Y, and Z are substituted into the equations, then S_{xa}, S_{ya}, S_{za}, the components acting upon the member ab at end a, will have directions consistent with the directions of the references axes x, y, and z. The stress S in the member ab as obtained from equation 181 will be tension when positive and compression when negative. It is therefore apparent that the correct signs for all stresses are obtained automatically if the quantities X, Y, and Z are consistent with the coordinates of the joints.

Example 61. The stresses for all members of the space frame in Fig. 171 will be determined by means of the deformation equations.

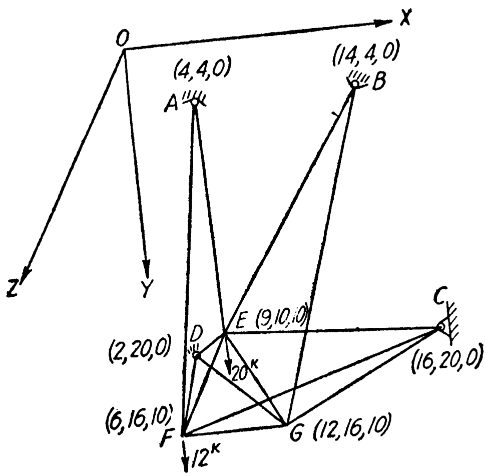

Fig. 171.

As points A, B, C, and D are considered fixed supports, only the displacements of joints E, F, and G are involved in the analysis. The necessary constants in terms of Q, X, Y, and Z are tabulated in Table 14 for each joint. It should be noted that the sign of the terms X, Y, and Z is important. The quantities from Table 14 are then substituted into the compatibility equations 184a, b, and c, which are written for each joint E, F, and G. This operation gives equations 185 which are solved for the displacements u, v, and w for each joint. Various meth-

ods of solving such simultaneous equations are possible, but, as each person has his favorite procedure, no particular solution will be given here. The numerical values of the stresses given in Table 15 are then obtained by substituting the numerical value of the displacements in equation 181. The components S_x, S_y, and S_z are obtained by multiplying the stress by the ratios $\frac{X}{L}$, $\frac{Y}{L}$, and $\frac{Z}{L}$. The discrepancies in the summations of the X, Y, and Z components are due to solution of the simultaneous equations by successive approximations. However, the results are sufficiently accurate for all practical purposes.

TABLE 14

Member	A in.^{2}	L ft	$Q = \frac{A}{L^3}$ $\times 10^{-2}$	QX^2	QY^2	QZ^2	QXY	QXZ	QYZ
EA	5	12.68	0.245	6.13	8.81	24.5	7.35	12.25	14.70
EB	5	12.68	0.245	6.13	8.81	24.5	−7.35	−12.25	14.70
EC	8	15.78	0.204	10.0	20.4	20.4	14.28	−14.28	−20.4
ED	8	15.78	0.204	10.0	20.4	20.4	−14.28	14.28	−20.4
EF	3	6.70	0.997	8.98	35.95	0	−17.97	0	0
EG	3	6.70	0.997	8.98	35.95	0	17.97	0	0
			Total	50.22	130.32	89.8	0	0	−11.4
FA	5	15.76	0.128	0.51	18.41	12.80	3.07	2.56	15.35
FC	3	14.7	0.095	9.46	1.51	9.46	3.79	−9.46	−3.79
FD	3	11.49	0.198	3.17	3.17	19.80	−3.17	7.92	−7.92
FE	3	6.7	0.997	8.98	35.95	0	−17.97	0	0
FG	3	6.0	1.39	50.0	0	0	0	0	0
			Total	72.12	59.04	42.06	−14.28	1.02	3.64
GB	5	15.76	0.128	0.51	18.41	12.80	−3.07	−2.56	15.35
GC	3	11.49	0.198	3.17	3.17	19.80	3.17	−7.92	−7.92
GD	3	14.7	0.095	9.46	1.51	9.46	−3.79	9.46	−3.79
GE	3	6.7	0.997	8.98	35.95	0	17.97	0	0
GF	3	6.0	1.39	50.0	0	0	0	0	0
			Total	72.12	59.04	42.06	14.28	−1.02	3.64

JOINT E

$$50.22u_e - 8.98u_f - 8.98u_g + 17.97v_f - 17.97v_g = 0$$
$$130.3v_e - 11.4w_e + 17.97u_f - 17.97u_g - 35.95v_f - 35.95v_g = 20{,}000 \text{ lb}$$
$$-11.4v_e + 89.8w_e = 0 \quad \text{or} \quad w_e = 0.127v_e$$

JOINT F

$$72.12u_f - 14.28v_f + 1.02w_f - 8.98u_e - 50.0u_g + 17.97v_e = 0$$
$$-14.28u_f + 59.04v_f + 3.64w_f + 17.97u_e - 35.95v_e = 12{,}000 \text{ lb}$$
$$1.02u_f + 3.64v_f + 42.06w_f = 0 \quad \text{or} \quad w_f = -0.024u_f - 0.0865v_f \tag{185}$$

JOINT G

$$72.12u_g + 14.28v_g - 1.02w_g - 8.98u_e - 50.0u_f - 17.97v_e = 0$$
$$14.28u_g + 59.04v_g + 3.64w_g - 17.97u_e - 35.95v_e = 0$$
$$-1.02u_g + 3.64v_f + 42.06w_g = 0 \quad \text{or} \quad w_g = 0.024u_g - 0.0865v_g$$

Solving the above equations, we obtain

$$u_e = -81.1 \qquad u_f = 39.5 \qquad u_g = 66.4$$
$$v_e = 323.8 \qquad v_f = 436.1 \qquad v_g = 156.3$$
$$w_e = 41.1 \qquad w_f = -38.7 \qquad w_g = -11.9$$

Substituting these values back in equations 181 and 182 gives the stresses shown in Table 15

TABLE 15

MEMBER	S	L	S_x	S_y	S_z
EA	+6,040	12.68	+2,390	+2,860	+4,770
EB	+8,550	12.68	−3,380	+4,050	+6,750
EC	−7,250	15.78	+3,220	+4,600	−4,600
ED	−10,920	15.78	−4,850	+6,920	−6,920
EF	+2,090	6.70	+940	−1,870	0
EG	−3,760	6.70	+1,690	+3,370	0
		Total	+10	+19,930	0
FA	+9,920	15.76	+1,260	+7,560	+6,300
FC	−3,520	14.7	+2,400	+960	−2,400
FD	−4,480	11.49	−1,570	+1,570	−3,910
FE	+2,090	6.7	−940	+1,870	0
FG	+1,348	6.0	−1,350	0	0
		Total	−200	+11,960	−10
GB	+3,280	15.76	−420	+2,500	+2,080
GC	−2,040	11.49	+710	+710	−1,780
GD	−162	14.7	−110	+40	−110
GE	−3,760	6.7	−1,690	−3,360	0
GF	+1,348	6.0	+1,350	0	0
		Total	−160	−110	+190

105. Group and Rigid Block Displacements. In the preceding discussion and application of the deformation equations it was assumed that the movement of the joints was governed only by equations 184*a*, *b*, and *c*. However, the conditions of the problem will frequently permit a simplification of the equations by assuming certain relations between the relative movement of the joints. Such assumptions must, of course, be based upon an estimation of the actual physical conditions that exist in the structure and will probably never provide an exact solution.

Nevertheless, such assumptions will usually be necessary to make the solution possible from a practical viewpoint, and the results will, in general, satisfy the equilibrium conditions. Any local discrepancy can be distributed later without introducing any serious errors.

The assumption of rigid block displacements will be illustrated by the solution of the structure in Fig. 172. An exact analysis for all stresses by considering the movement of every joint would be impractical. However, if the portion above section x–x is assumed to move as a rigid

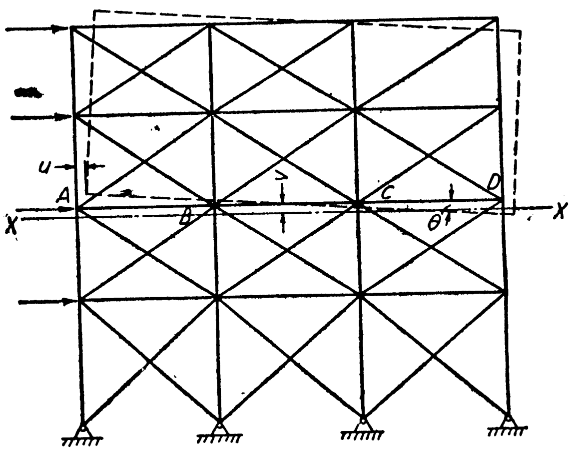

FIG. 172.

body, then the movement of points A, B, C, and D can be expressed in terms of one vertical displacement v, a horizontal displacement u, and a rotation θ. These displacements must satisfy the three equilibrium conditions for the rigid body, and consequently the problem reduces to three equations with three unknowns. Other sections can then be taken and various sets of stresses obtained that satisfy the equilibrium condition for the structure as a unit. Local discrepancies in the equilibrium conditions at a joint, or in the strain relations, must be corrected by approximation. The success of such an analysis will depend upon the astuteness of the person making the calculations, for the procedure must be varied to fit the conditions of the problem.

Instead of relating the movements of the various joints by assuming a rigid body condition, it will sometimes be better to assume other types of related displacements which can be combined to satisfy the compatibility equations. Such solutions will usually be involved, but are often the most direct and accurate method of analysis for complicated space frameworks. The reader should consult the references at the end

of the chapter, particularly Ref. 10·7, for applications of this method to such problems as the hull stresses in rigid airships.

106. Stresses in Engine Mount Frames. The stress in any member of a supporting frame for a radial aircraft engine (see Fig. 173) can be readily obtained by assuming that the engine and ring move as a rigid body. The analysis is readily made by means of the following sequence of steps in the numerical solution.*

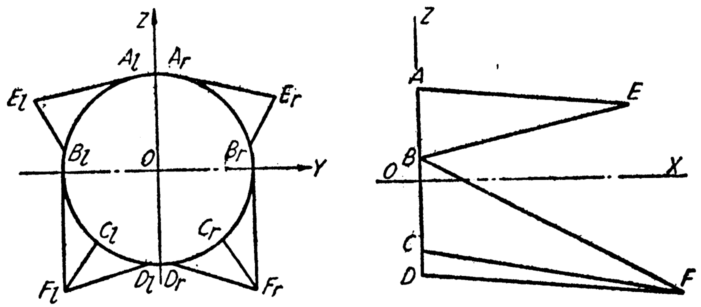

FIG. 173.

1. Tabulate coordinates x, y, and z for all points with respect to O. Prepare tables X, Y, Z, L, A, Q, QX^2, QY^2, QZ^2, QXY, QXZ, and QYZ. Assume that all points on the ring mount must move as a rigid body; that is, they can have a movement along the x axis equal to u, along the y axis equal to v, along the z axis equal to w, a rotation about the y axis equal to θ_y, about the z axis equal to θ_z, and about the x axis equal to ϕ_x (rotation).

2. Assume any displacement $+u$ for all joints A, B, C, and D, holding E and F fixed. Calculate the stress S in each member and the summations of S_x, S_y, and S_z, respectively, at each support. Determine the necessary applied force at O from the S values, and represent the various components by P_{x1}, P_{y1}, P_{z1}, M_{x1}, M_{y1}, and M_{z1}.

3. Assume any displacement $+v$ for the joints A, B, C, and D, and do the same as in part 2 above. Determine the components of the applied force at O, P_{x2}, P_{y2}, P_{z2}, M_{x2}, M_{y2}, and M_{z2}.

4. Assume any displacement $+w$. Calculate stresses S and the components of the applied force at O, P_{x3}, P_{y3}, P_{z3}, M_{x3}, M_{y3}, and M_{z3}.

5. Assume a rotation ϕ_x, and calculate stresses S and the components P_{x4}, P_{y4}, P_{z4}, M_{x4}, M_{y4}, and M_{z4}.

* A similar arrangement for the solution of this problem has been proposed and used by Dr. R. B. Moorman, Goodyear Aircraft Corp., Akron, Ohio.

6. For an assumed rotation θ_y, calculate stress and the necessary components at point O, P_{x5}, P_{y5}, P_{z5}, M_{z5}, M_{y5}, and M_{z5}.

7. Also, assume any rotation θ_z, calculate stresses and components of the applied load at point O, P_{x6}, P_{y6}, P_{z6}, M_{x6}, M_{y6}, and M_{z6}.

To obtain the maximum combined stress in any member for any loading condition, the following separate problems should be solved.

A. Calculate the stresses for an axial load F'_x of 1000 lb applied at O. To solve this problem, we can set

$$\begin{aligned}
P_{x1} + P_{x2} + P_{x3} + P_{x4} + P_{x5} + P_{x6} &= F'_x = 1000 \\
P_{y1} + P_{y2} + P_{y3} + P_{y4} + P_{y5} + P_{y6} &= 0 \\
P_{z1} + P_{z2} + P_{z3} + P_{z4} + P_{z5} + P_{z6} &= 0 \\
M_{x1} + M_{x2} + M_{x3} + M_{x4} + M_{x5} + M_{x6} &= 0 \\
M_{y1} + M_{y2} + M_{y3} + M_{y4} + M_{y5} + M_{y6} &= 0 \\
M_{z1} + M_{z2} + M_{z3} + M_{z4} + M_{z5} + M_{z6} &= 0
\end{aligned} \tag{186}$$

Calculate u, v, w, ϕ_x, θ_y, and θ_z, and solve for all stresses. Designate stresses by S_A.

B. Calculate the stresses for a horizontal load F'_y of 1000 lb applied at O. Let stresses be equal to S_B. To solve this problem, the following conditions must be satisfied:

$$\begin{aligned}
P_{x1} + P_{x2} + P_{x3} + P_{x4} + P_{x5} + P_{x6} &= 0 \\
P_{y1} + P_{y2} + P_{y3} + P_{y4} + P_{y5} + P_{y6} &= 1000 \\
P_{z1} + P_{z2} + P_{z3} + P_{z4} + P_{z5} + P_{z6} &= 0 \\
M_{x1} + M_{x2} + M_{x3} + M_{x4} + M_{x5} + M_{x6} &= 0 \\
M_{y1} + M_{y2} + M_{y3} + M_{y4} + M_{y5} + M_{y6} &= 0 \\
M_{z1} + M_{z2} + M_{z3} + M_{z4} + M_{z5} + M_{z6} &= 0
\end{aligned} \tag{187}$$

C. Calculate the stresses for a vertical load F'_z of 1000 lb applied at O. Let stresses be S_C.

D. Calculate the stresses for a torque M_x of 1000 ft-lb. Let stresses be S_D.

E. Calculate stresses for a moment M_y of 1000 ft-lb. Let stresses be S_E.

F. Calculate the stresses for a moment M_z of 1000 ft-lb. Let stresses be S_F.

Then, for any combination of applied forces, F_x, F_y, F_z, M_x, M_y, and M_z, the stress S in any member is

$$S = \frac{F_xS_A + F_yS_B + F_zS_C + M_xS_D + M_yS_E + M_zS_F}{1000} \tag{188}$$

Shearing Stresses in Thin-Walled Closed Sections

107. Shearing Stress Due to Torsion. The shearing stress f_s on any element ds of a thin-walled closed section, Fig. 174, due to a twisting moment T is assumed to be uniformly distributed over the thickness t in the plane of the cross section. The shearing force q for ds equal to unity is therefore equal to

$$q = f_st \text{ pounds per linear inch} \tag{189}$$

where q is defined as the unit shearing force and the distribution of q around the perimeter of a closed section is termed the shear flow. The value of q in a single cell closed section subjected to torsion only must be constant around the section regardless of the thickness t unless the shearing deformation is prevented. This statement follows from the fact that the unit shearing force on an element such as $abcd$ in Fig. 174 can vary only if there is a change in the normal stress, or if some external shearing stress is applied to the surface. As these axial or external stresses are not acting when the section is subjected to torsion only, the shearing force q must be constant for all elements.

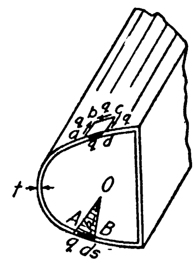

FIG. 174.

The resisting torque about any point O due to any element whose length is ds is

$$dT = (q\,ds)h = 2q(\text{area } OAB) \tag{190}$$

The total resisting torque T_r which must be equal to the applied torque T is, therefore,

$$T_r = T = \oint qh\,ds = 2qA$$

where A is the total area enclosed by the section and $\oint$ represents integration around the perimeter. The constant unit shearing force q is therefore easily calculated by the following expression, usually called Bredt's formula,

$$q = \frac{T}{2A} \tag{191}$$

108. Shearing Deformation Due to Torsion. If a cut is made at any section x–x in a closed section, Fig. 175, the shearing deformation of all elements will tend to produce a relative movement of the two sides. The effect of the shearing stress upon one element is shown in Fig. 175. The movement of points a and b for a unit of length ad and unit perimeter ab is

$$\delta_s = \frac{f_s}{G} = \frac{q}{tG} \tag{192}$$

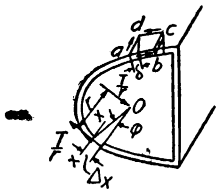

FIG. 175.

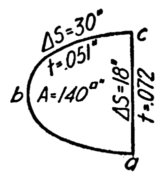

FIG. 176.

The internal work or strain energy in the material for any distance ds along the perimeter is therefore

$$d\,W_i = \frac{q}{2}\,\delta_s\,ds = \frac{Tq\,ds}{4AtG}$$

If the torque about any fixed point O is T, it can be replaced by the couple $\dfrac{T}{r}$ as shown in Fig. 175. The external work per unit of length is therefore equal to

$$\frac{T}{2r}\,\Delta_x = \frac{T\phi}{2}$$

Equating the external work to the total internal work per unit length of cell, we obtain

$$\frac{T\phi}{2} = \frac{T}{4AG}\oint \frac{q\,ds}{t}$$

or

$$\phi = \frac{1}{2AG}\oint \frac{q\,ds}{t} \tag{193a}$$

where ϕ is the angle of twist per unit length of cell.

In general the perimeter is divided into several elements whose lengths are known and over which the shear flow q and thickness t are constant. The integration should then be replaced by the summation of the terms for the elements, that is,

$$\phi = \frac{1}{2AG}\sum \frac{q\,\Delta_s}{t} \tag{193b}$$

Example 62. The shearing stresses and the angle of twist will be calculated for the cell in Fig. 176 for a torque of 30,000 in.-lb.

$$q = \frac{30{,}000}{(2)(140)} = 107 \text{ lb per in.}$$

For nose section

$$f_s = \frac{107}{0.051} = 2100 \text{ lb per in.}^2$$

For web

$$f_s = \frac{107}{0.072} = 1490 \text{ lb per in.}^2$$

From equation 193*b*,

$$\phi = \frac{107}{(2)(140)G}\left(\frac{30}{0.051} + \frac{18}{0.072}\right) = \frac{320}{G} \text{ radians per inch of length}$$

109. Torsional Stresses in Multiple-Cell Sections. The shearing stresses in the walls of multiple-cell sections such as Fig. 177 are

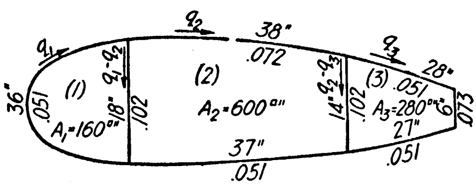

FIG. 177.

calculated by means of equations 191 and 193 together with the following conditions:

(*a*) The shear flow q' into an intersection of elements must equal the shear flow out, or $q' - q'' - q''' = 0$ (see Fig. 177*a*).

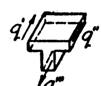

FIG. 177*a*.

(*b*) The angle of twist ϕ for each cell is the same.

(*c*) The sum of the resisting torques for all cells must equal the total torque applied to the section.

Condition (*a*) is based upon the equilibrium of a corner element with respect to the axial direction. Taking

$$\Sigma F_x = 0$$

$$q' - q'' - q''' = 0 \tag{194}$$

or

$$q''' = q' - q''$$

The second condition (*b*) can be visualized by considering the movement of a gap in the element common to two cells, with respect to the same point O. By referring to Fig. 175 it is apparent that

$$\phi_1 = \phi_2$$

and

$$\phi_2 = \phi_3$$

where

$$\begin{aligned}\phi_1 &= \frac{1}{2A_1G}\sum\frac{q\,\Delta s}{t}\\ \phi_2 &= \frac{1}{2A_2G}\sum\frac{q\,\Delta s}{t}\\ \phi_3 &= \frac{1}{2A_3G}\sum\frac{q\,\Delta s}{t}\end{aligned} \tag{195}$$

The third condition (*c*) states that

$$T_1 + T_2 + T_3 = T$$

or

$$2(q_1A_1 + q_2A_2 + q_3A_3) = T \tag{196}$$

As the shear flow can be expressed in terms of one unknown q value for each cell, the above relations are sufficient to obtain a unique solution.

Example 63. The shearing stresses in the three-cell section shown in Fig. 177 will be calculated for a torque of 40,000 ft-lb. The notation for the shear flow in the various elements is indicated on the sketch. By means of equation 193*b* the angle of twist ϕ for each cell can be written in the form

$$\phi_1 = \frac{1}{(2)(160)G}\left[\frac{36q_1}{0.051} + \frac{18(q_1 - q_2)}{0.102}\right]$$

$$\phi_2 = \frac{1}{(2)(600)G}\left[\frac{38q_2}{0.072} + \frac{14(q_2 - q_3)}{0.102} + \frac{37q_2}{0.051} + \frac{18(-q_1 + q_2)}{0.102}\right]$$

$$\phi_3 = \frac{1}{(2)(280)G}\left[\frac{28q_3}{0.051} + \frac{6q_3}{0.072} + \frac{27q_3}{0.051} + \frac{14(-q_2 + q_3)}{0.102}\right]$$

Equating

$$\phi_1 = \phi_2 = \phi_3$$

we obtain

$$\begin{aligned}2.76q_1 - 0.552q_2 = -0.147q_1 + 1.305q_2 - 0.114q_3 - 0.147q_1\\ + 1.305q_2 - 0.114q_3 = -0.245q_2 + 2.31q_3\end{aligned}$$

Eliminating q_3 from the above equations gives

$$q_2 = 1.625q_1$$

Eliminating q_2 from the above equations gives

$$q_3 = 0.977q_1$$

Substituting these values of q_3 and q_2 into equation 196 gives $2[160q_1 + 600\,(1.625q_1) + 280\,(0.977q_1)] = (40{,}000)(12)$, from which

$$q_1 = 170.5 \text{ lb per in.}$$

and, therefore,

$$q_2 = (1.625)(170.5) = 277 \text{ lb per in.}$$

$$q_3 = (0.978)(170.5) = 166.5 \text{ lb per in.}$$

110. Flexural Stresses in Unsymmetrical Closed Sections. The normal stresses on the effective area of an unsymmetrical closed section, as in Fig. 178, can be calculated by the usual flexure formula if the

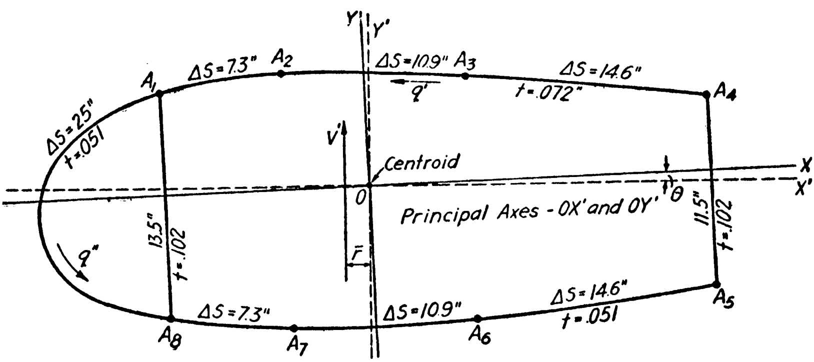

FIG. 178.

principal axes of inertia are used as the axes of rotation. The principal axes of inertia OX' and OY' are ordinarily determined with respect to any reference axes OX and OY passing through the centroid O of the effective area by means of the equation

$$\tan 2\theta = \frac{2I_{xy}}{I_y - I_x} \tag{197}$$

The principal moments of inertia are then calculated by equations 198*a* and *b*.

$$I_{x'} = I_x \cos^2\theta + I_y \sin^2\theta - I_{xy}\sin 2\theta \tag{198a}$$

$$I_{y'} = I_y \cos^2\theta + I_x \sin^2\theta + I_{xy}\sin 2\theta \tag{198b}$$

in which I_x and I_y are the moments of inertia about OX and OY, respectively. I_{xy} is the product of inertia about axes OX and OY.

The other necessary relations are

$$M_{x'} = M_x \cos\theta - M_y \sin\theta \tag{199a}$$

$$M_{y'} = M_y \cos\theta + M_x \sin\theta \tag{199b}$$

$$x' = x\cos\theta + y\sin\theta \tag{199c}$$

$$y' = y\cos\theta - x\sin\theta \tag{199d}$$

The normal stresses f are calculated for bending about the principal axes by the usual bending formula

$$f = \frac{M_{x'}y'}{I_{x'}} + \frac{M_{y'}x'}{I_{y'}} \tag{200}$$

However, the normal stress can also be calculated directly from the reference axes OX and OY by means of the following equation (see Ref. 10·9):

$$f = \frac{(M_yI_x - M_xI_{xy})x}{I_xI_y - I_{xy}^2} + \frac{(M_xI_y - M_yI_{xy})y}{I_xI_y - I_{xy}^2} \tag{201}$$

Example 64. The normal stresses acting upon the flange area A_1 in the closed section of Fig. 178 will be calculated by means of equations 198, 199, and 200. The properties of the section with respect to the axes OX and OY, where O is the centroid of the areas A_1, A_2, A_3 $\cdots$, are given in Table 16.

TABLE 16. PROPERTIES WITH RESPECT TO AXES OX AND OY

No.	Area in.^{2}	x	y	Ax^2	Ay^2	Axy
A_1	1.2	−12.1	+6.11	175.69	44.80	−88.72
A_2	0.6	−4.9	+6.81	14.41	27.83	−20.02
A_3	0.4	+5.9	+6.41	13.92	16.44	15.13
A_4	0.6	+20.3	+4.51	247.25	12.20	54.93
A_5	0.6	+20.3	−6.99	247.25	29.32	−85.14
A_6	0.4	+5.9	−8.29	13.92	27.49	−19.56
A_7	0.4	−4.9	−8.19	9.60	26.83	16.05
A_8	0.8	−12.1	−7.39	117.13	43.69	71.54
				I_y = 839.17	I_x = 228.60	I_{xy} = −55.79

From equation 197 the angle θ that the principal axis OX' makes with OX is

$$\tan 2\theta = \frac{(2)(-55.79)}{839.17 - 228.60} = -0.1827$$

$$2\theta = 169°\,38'\,40'' \quad \text{or} \quad 349°\,38'\,40''$$

$$\theta = 84°\,49'\,20'' \quad \text{or} \quad -5°\,10'\,40''$$

$$\cos - 5°\,10'\,40'' = 0.9959 \qquad \sin - 5°\,10'\,40'' = -0.0901$$

$$x' = 0.9959x - 0.0901y$$

$$y' = 0.9959y + 0.0901x$$

From equations 198a and b, the values of the principal moments of inertia are

$$I_{x'} = (228.60)(0.9959)^2 + (839.17)(0.0901)^2 - (-55.79)(-0.1764) = 223.70 \text{ in.}^4$$

$$I_{y'} = (839.17)(.9959)^2 + (228.60)(0.0901)^2 + (-55.79)(-0.1764) = 844.00 \text{ in.}^4$$

For area A_1

$$x' = (-12.1)(0.9959) + (6.11)(-0.0901) = -12.60$$

$$y' = (6.11)(0.9959) - (-12.1)(-0.0901) = 4.99$$

From equation 200,

$$f = \frac{M_{x'}(4.99)}{223.7} + \frac{M_{y'}(-12.60)}{844.0} = 0.0223M_{x'} - 0.0149M_{y'}$$

but

$$M_{x'} = 0.9959M_x + 0.0901M_y$$

$$M_{y'} = -0.0901M_x + 0.9959M_y$$

Therefore

$$f = 0.0235M_x - 0.0128M_y$$

The same result is obtained directly from equation 201 by substituting the numerical values of I_x, I_y, and I_{xy}.

111. Shearing Stresses in Closed Section Due to Transverse Forces. From the equilibrium conditions for an element $abcd$, Fig. 179, it is apparent that any variation in the normal stresses f due to a change in the bending moment must cause a change in the shearing stress. Thus, if f_1 and f_2 are the

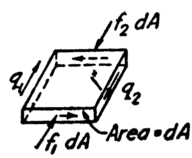

FIG. 179.

normal stresses and q_1 and q_2 are the shearing forces per unit length, then

$$\Sigma F_z = 0$$

or

$$(f_1 - f_2)\, dA + q_1 - q_2 = 0$$

$$q_2 = q_1 + (f_1 - f_2)\, dA$$

But

$$f_1 - f_2 = \frac{\Delta M_{x'} y'}{I_{x'}} + \frac{\Delta M_{y'} x'}{I_{y'}} = \frac{V_{y'} y'}{I_{x'}} + \frac{V_{x'} x'}{I_{y'}}$$

where

$$\Delta\, M_{x'} \text{ per unit length} = V_{y'} \cdot 1$$

$$\Delta\, M_{y'} \text{ per unit length} = V_{x'} \cdot 1$$

$V_{y'}$ and $V_{x'}$ being the transverse shear parallel to the y' and x' axes, respectively.

Let

$$Q_{x'} = y'\, dA$$

$$Q_{y'} = x'\, dA$$

then

$$q_2 = q_1 + \frac{V_{y'} Q_{x'}}{I_{x'}} + \frac{V_{x'} Q_{y'}}{I_{y'}} \tag{202}$$

Equation 202 can be stated in the following terms: **the unit shearing force q_2 at any point 2 is equal to the unit shearing force q_1 at any point 1, plus the change in the unit shearing force $\frac{VQ}{I}$ between the two points with respect to the principal axes.** If some point where the shearing stress is zero is known, such as at a free surface or an axis of symmetry, then all shearing stresses can be calculated from the change in the unit shearing force

$$q = \frac{V_{y'} Q_{x'}}{I_{x'}} + \frac{V_{x'} Q_{y'}}{I_{y'}} \tag{203}$$

However, for a closed section, such as Fig. 178, there are no points at which the shearing stresses are known, and, consequently, there is one redundant q value in each cell. If the unit shearing force in the sheet between the flange areas A_2 and A_3, Fig. 178, is designated by q' and the unit shearing force in the nose section by q'', then the unit shearing force q in any part of the perimeter can be expressed in terms of q' and q'' by equation 202. The two redundant quantities q' and q''

can then be determined from the condition that no twisting of the section takes place, in other words, that the angle of twist ϕ for each cell is zero. This strain condition gives for each cell in the section an equation of the type

$$\phi = \frac{1}{2AG} \sum q \frac{\Delta s}{t} = 0$$

or simply

$$\sum \frac{q\,\Delta s}{t} = 0 \tag{204}$$

By applying equation 204 to each cell, the two redundant quantities q' and q'' can be calculated. With q' and q'' known, the shearing force q at any part of the perimeter is determined from equation 202. This method of solution is applicable to any number of cells. The numerical operations will be explained by an example.

Example 65. The shear flow around the section shown in Fig. 178 will be determined for a transverse shear $V_{y'}$ of 1000 lb parallel to the y' axis. The unit shearing force q' between the flange areas A_2 and A_3 and q'' in the nose section are selected as the redundant quantities. Using the value of $I'_{x'}$ that was computed in Example 64, the unit shearing force q at any part of the perimeter in the rear cell is given by equation 202 as

$$q = q' + \frac{1000Ay'}{223.7}$$

As the flange areas are considered concentrated at certain points, the shear flow q is constant between these areas. Table 17 gives the values of q for each segment of the perimeter between the flange areas. The calculations were made by starting with q' between areas A_2 and A_3 and progressing clockwise around the cell.

TABLE 17

Segment	Flange Area	y'	$Q = Ay'$	$\Delta q = \frac{1000Q}{223.7}$	$q = q' + \Delta q$
					q'
A_2–A_3	0.6	+6.34	+3.80	16.99	$q' + 12.34$
A_3–A_4	0.4	+6.91	2.76	12.34	$q' + 29.28$
A_4–A_5	0.6	+6.32	3.79	16.94	$q' + 15.51$
A_5–A_6	0.6	−5.13	−3.08	−13.77	$q' + 1.7$
A_6–A_7	0.4	−7.73	−3.09	−13.81	$q' - 13.68$
A_7–A_8	0.4	−8.60	−3.44	−15.38	$q' - q'' - 43.90$
A_8–A_1 (web)	0.8	−8.45	−6.76	−30.22	$q' - 17.08$
A_1–A_2	1.2	+5.00	+6.00	+26.82	

When the area A_8 is reached at the junction of the two cells the equilibrium condition requires that the shear flow out of the junction must equal the shear flow in plus the change in the unit shear -30.22. The shear flow in the web between A_8 and A_1 is therefore

$$q = q' - 13.68 - q'' - 30.22 = q' - q'' - 43.9$$

In the same manner the shear flow between areas A_1 and A_2 is

$$q = (q' - q'' - 43.9) + q'' + 26.82 = q' - 17.08$$

To evaluate q' and q'' the summation of the quantities $\frac{q\,\Delta s}{t}$ must be made zero for each cell. Table 18 gives the values of these quantities for the segments of the rear cell.

TABLE 18

Segment	q	Δs	t	$\frac{\Delta s}{t}$	$\frac{q\,\Delta s}{t}$
A_2–A_3	q'	10.9	0.072	151.3	$151.3q'$
A_3–A_4	$q' + 12.34$	14.6	0.072	202.7	$202.7q' + 2501.3$
A_4–A_5	$q' + 29.28$	11.5	0.102	112.7	$112.7q' + 3299.9$
A_5–A_6	$q' + 15.51$	14.6	0.051	286.2	$286.2q' + 4439.0$
A_6–A_7	$q' + 1.7$	10.9	0.051	213.6	$213.6q' + 363.1$
A_7–A_8	$q' - 13.68$	7.3	0.051	143.1	$143.1q' - 1957.6$
A_8–A_1	$q'-q'' - 43.90$	13.5	0.102	132.4	$132.4q' - 5812.4 - 132.4q''$
A_1–A_2	$q' - 17.08$	7.3	0.072	101.4	$101.4q' - 1731.9$

$$\text{For rear cell} \sum \frac{q\,\Delta s}{t} = 1343.4q' - 132.4q'' + 1101.4$$

$$\text{Nose section}\ \frac{25q''}{0.051} = 490.2q''$$

$$\text{Web section} = -132.4q' + 132.4q'' + 5812.4$$

or for front cell

$$\sum \frac{q\,\Delta s}{t} = -132.4q' + 622.6q'' + 5812.4$$

Equating the summation of the $\frac{q\,\Delta s}{t}$ terms for each cell to zero gives

$$1343.4q' - 132.4q'' + 1101.4 = 0$$

$$-132.4q' + 622.6q'' + 5812.4 = 0$$

from which we obtain

$$q' = -1.76 \text{ lb per in.} \qquad q'' = -9.72 \text{ lb per in.}$$

112. Shear Center for Closed Section. In the preceding discussion of the shear flow in a closed section due to transverse shear, the location of the resultant applied shearing force was not specified. As the determination of the redundant internal unit shearing forces is based on the assumption that there is *no twist* of the cross section due to the bending moments and transverse shears, the external shearing forces must be applied at that **point in the cross section about which the resultant internal twisting moment of the shearing forces around the perimeter is zero.** In algebraic terms this condition can be expressed as

$$\Sigma(q\ \Delta s)r' = 0 \tag{205}$$

where r' is the arm of the internal shearing force with respect to the shear center O'.

The position of the shear center O' is most easily obtained from the centroid O by the equation

$$V\bar{r} = \Sigma(q\ \Delta s)r \tag{206}$$

in which r and $\bar{r}$ are measured with respect to the centroid O.

The transverse shear can first be taken parallel to the Y' axis as in Example 65, and the distance $\bar{r}_1$ can be determined. The calculations should then be repeated for a shear parallel to the X' axis and the distance $\bar{r}_2$ should be determined. It follows from the definition of the shear center that any system of applied forces can be resolved into resultant shearing forces through the shear center, a resultant torque, and bending moments about the principal axes. The stresses due to these resultant forces can be analyzed independently in the manner explained in this section unless certain restrictions to the distribution of the stresses are imposed by the reactions or supports. Reference 10·11 should be consulted for a discussion of special problems that arise when the usual shearing deformation cannot take place.

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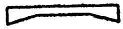
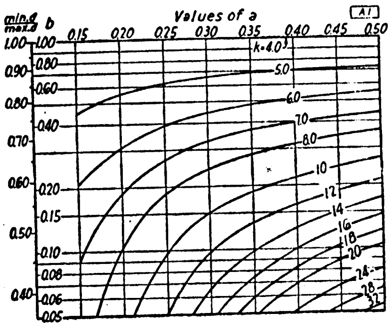
APPENDIX

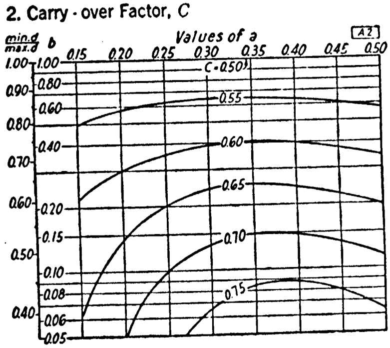
Coefficients and Fixed-End Moments for Symmetrical Beams with Straight Haunches



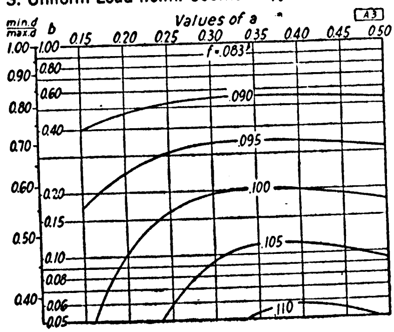
1. Stiffness Coefficient, k



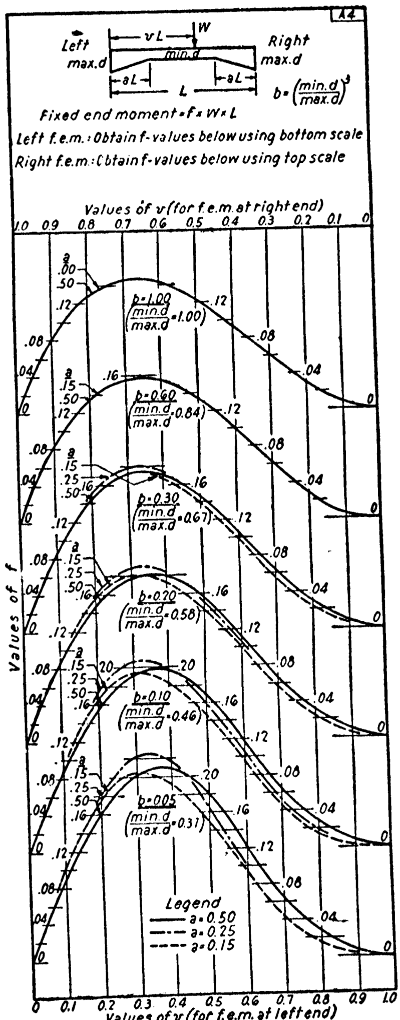
2. Carry-over Factor, C



3. Uniform Load f.e.m. Coefficient, f

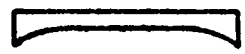


4. Concentrated Load f.e.m. Coefficient, f

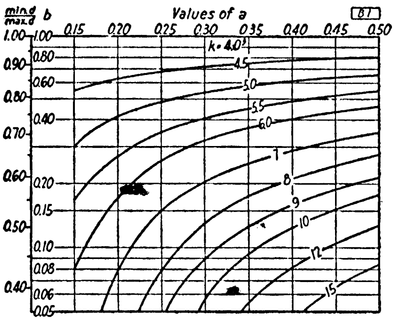


Courtesy of the Portland Cement Association.

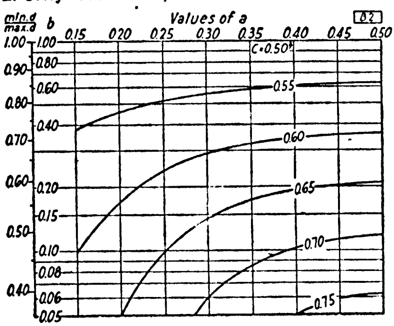
Coefficients and Fixed-End Moments for Symmetrical Beams with Parabolic Haunches



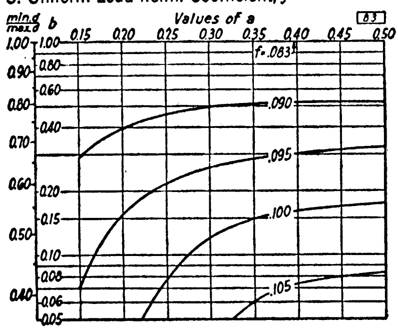
1. Stiffness Coefficient, k



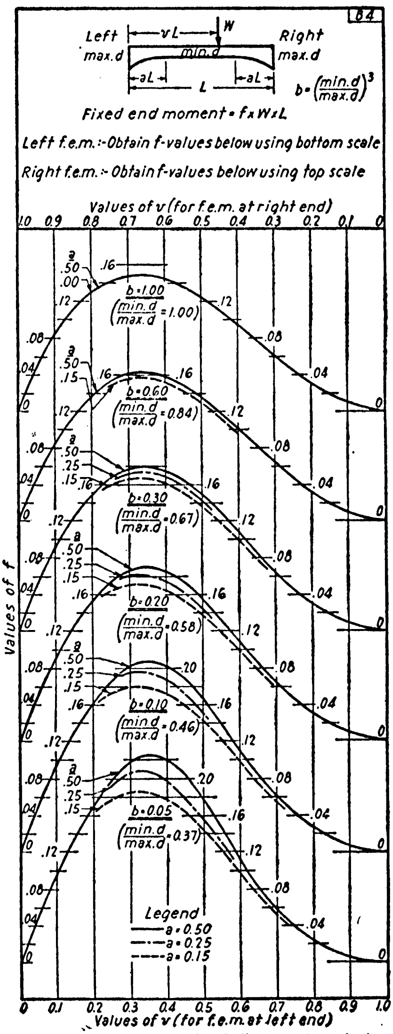
2. Carry-over Factor, C



3. Uniform Load f.e.m. Coefficient, f



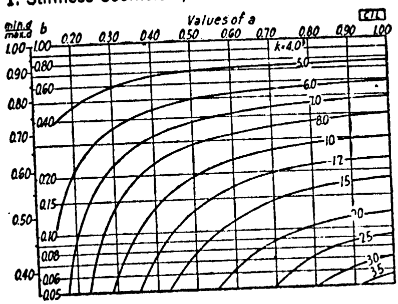
4. Concentrated Load f.e.m. Coefficient, f



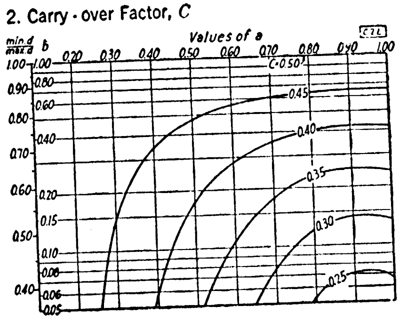
Courtesy of the Portland Cement Association.

Coefficients and Fixed-End Moments at the Large End of an Unsymmetrical Beam with a Straight Haunch

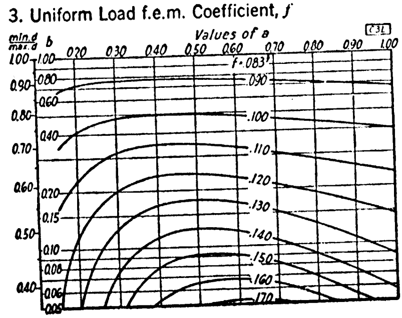
1. Stiffness Coefficient, k



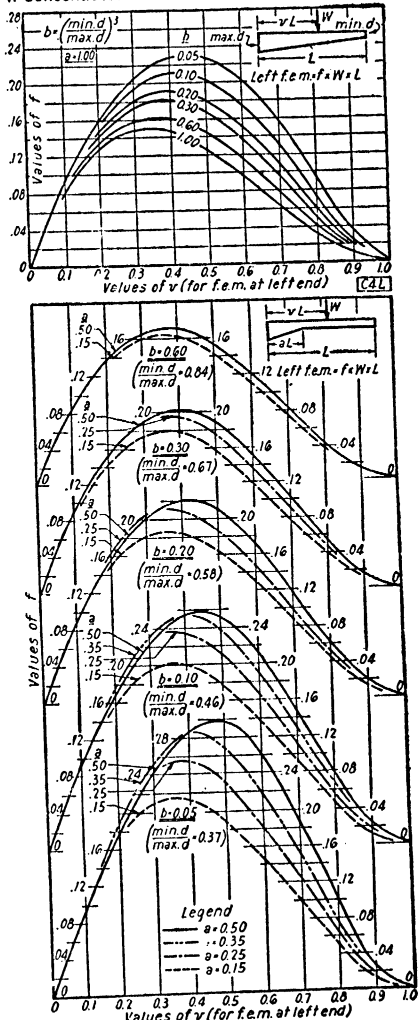
2. Carry-over Factor, C



3. Uniform Load f.e.m. Coefficient, f



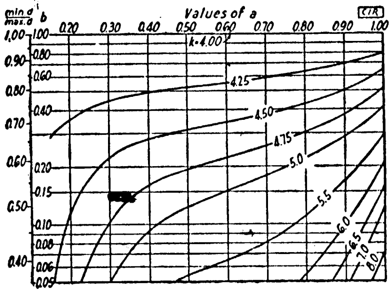
4. Concentrated Load f.e.m. Coefficient, f



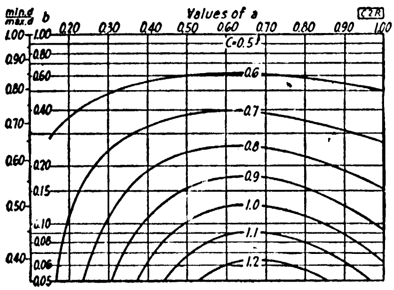
Courtesy of the Portland Cement Association.

Coefficients and Fixed-End Moments at the Small End of an Unsymmetrical Beam with a Straight Haunch

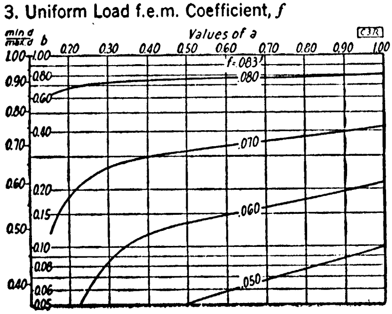
1. Stiffness Coefficient, k



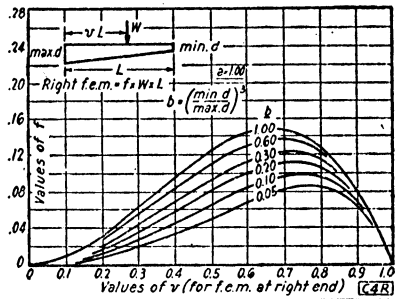
2. Carry-over Factor, C

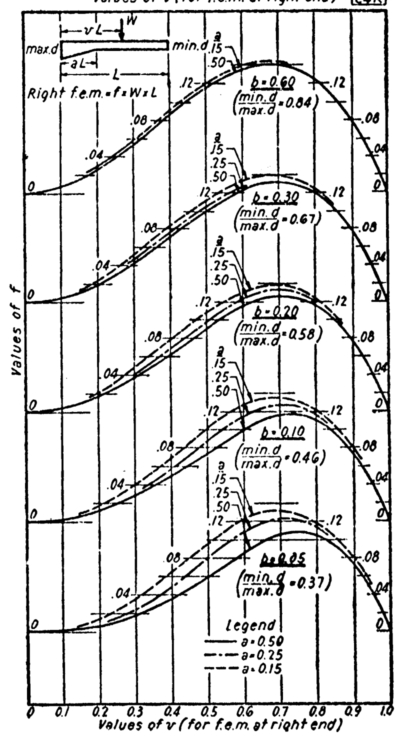


3. Uniform Load f.e.m. Coefficient, f



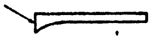
4. Concentrated Load f.e.m. Coefficient, f



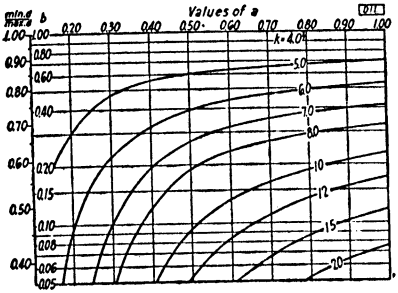


Courtesy of the Portland Cement Association.

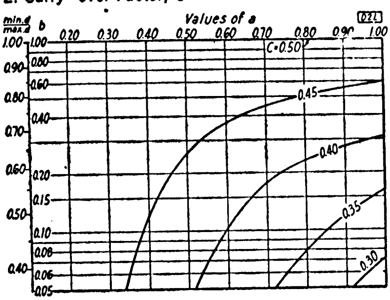
Coefficients and Fixed-End Moments at the Large End of an Unsymmetrical Beam with a Parabolic Haunch



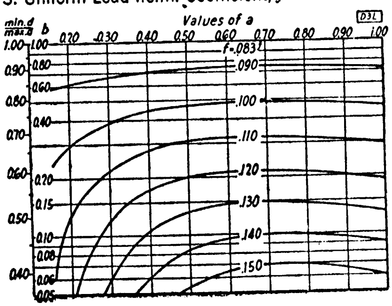
1. Stiffness Coefficient, k



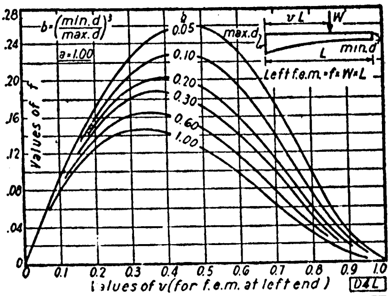
2. Carry - over Factor, C

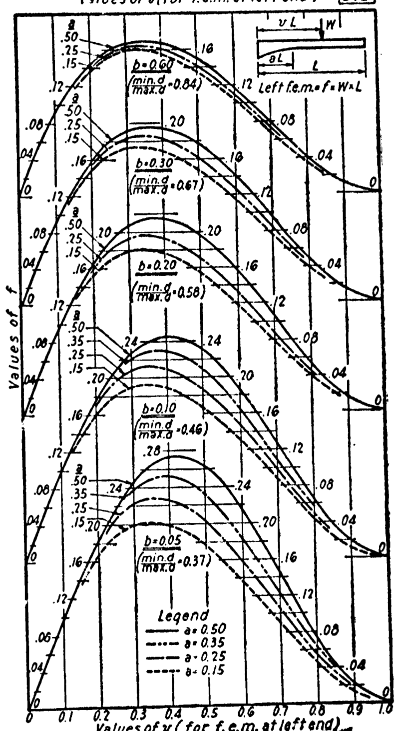


3. Uniform Load f.e.m. Coefficient, f



4. Concentrated Load f.e.m. Coefficient, f

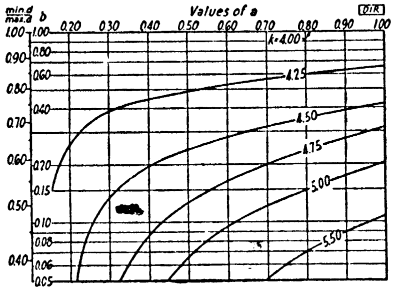




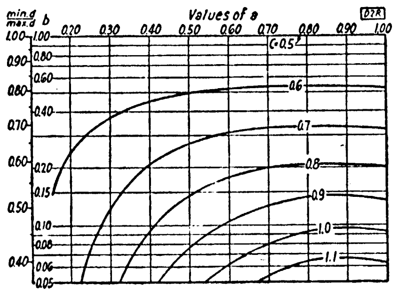
Courtesy of the Portland Cement Association.

Coefficients and Fixed-End Moments at the Small End of an Unsymmetrical Beam with a Parabolic Haunch

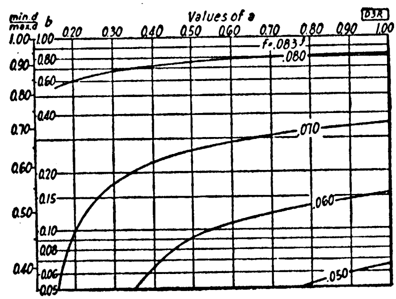
1. Stiffness Coefficient, k



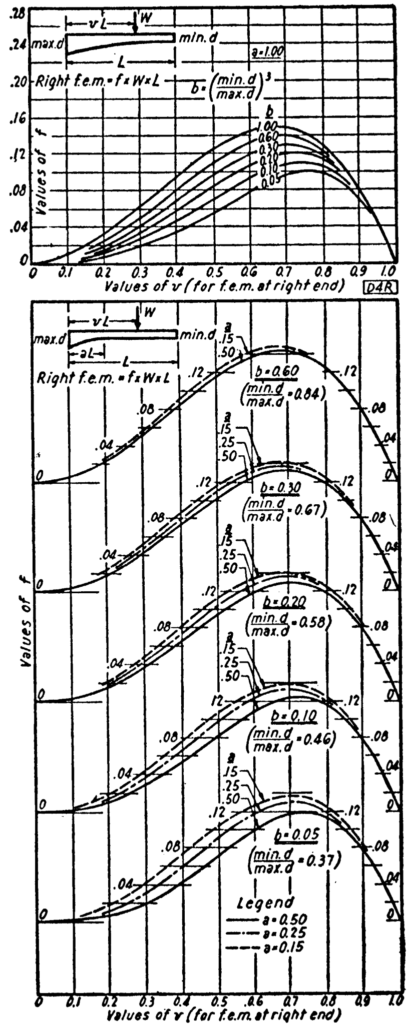
2. Carry-over Factor, C



3. Uniform Load f.e.m. Coefficient, f

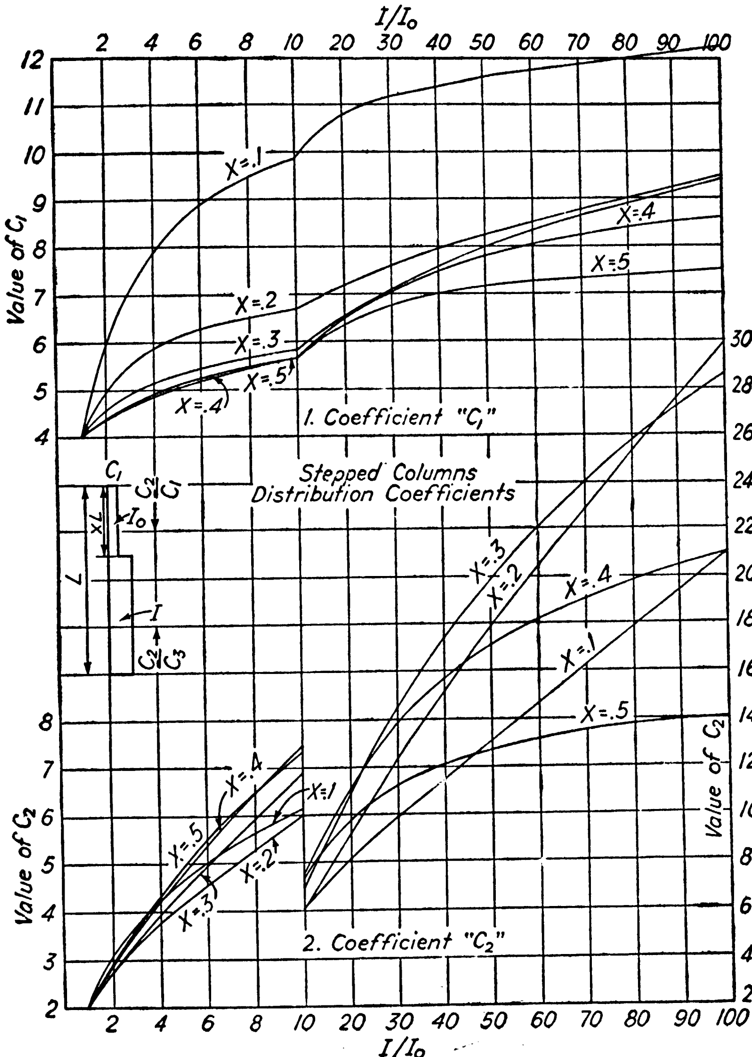


4. Concentrated Load f.e.m. Coefficient, f



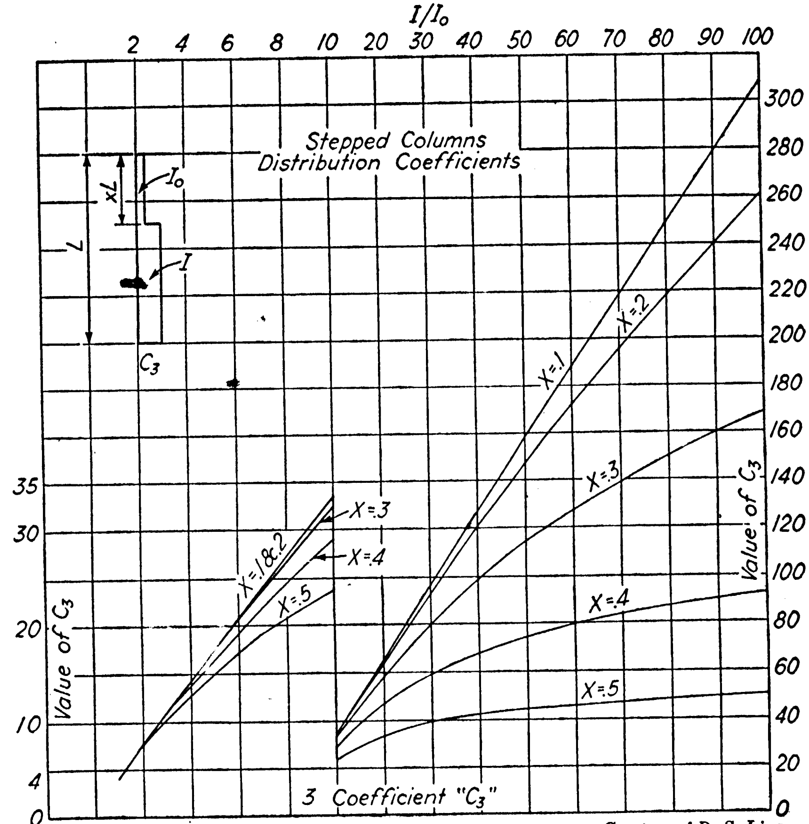
Courtesy of the Portland Cement Association.

COEFFICIENTS C_1 AND C_2 FOR BEAMS AND COLUMNS WITH SUDDEN CHANGE IN CROSS SECTION



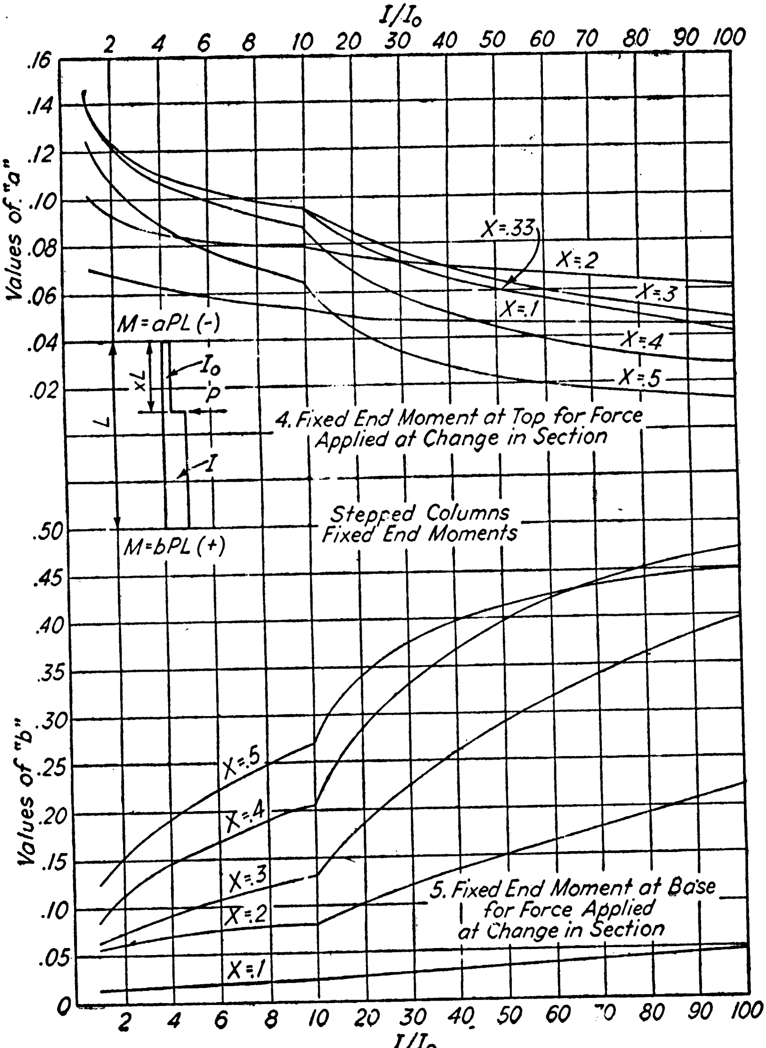
Courtesy of D. S. Ling.

COEFFICIENTS C_3 FOR BEAMS AND COLUMNS WITH SUDDEN CHANGE IN CROSS SECTION



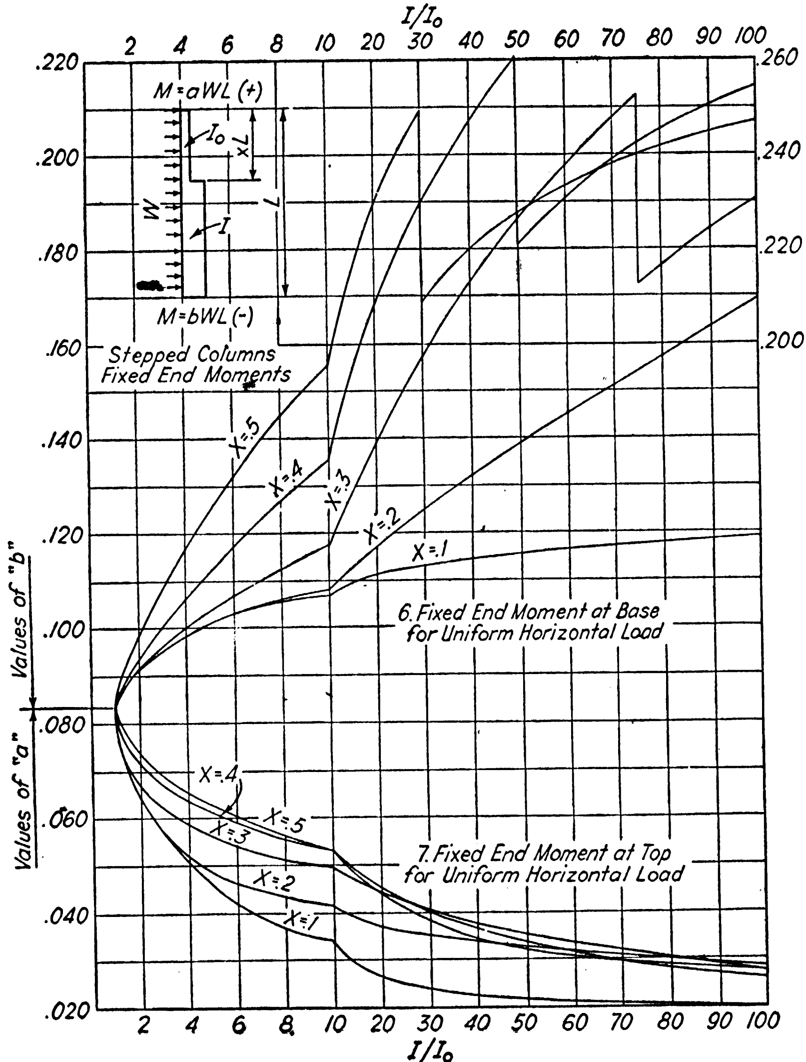
Courtesy of D. S. Ling.

FIXED-END MOMENTS FOR BEAMS AND COLUMNS WITH CONCENTRATED LOAD APPLIED AT CHANGE IN CROSS SECTION



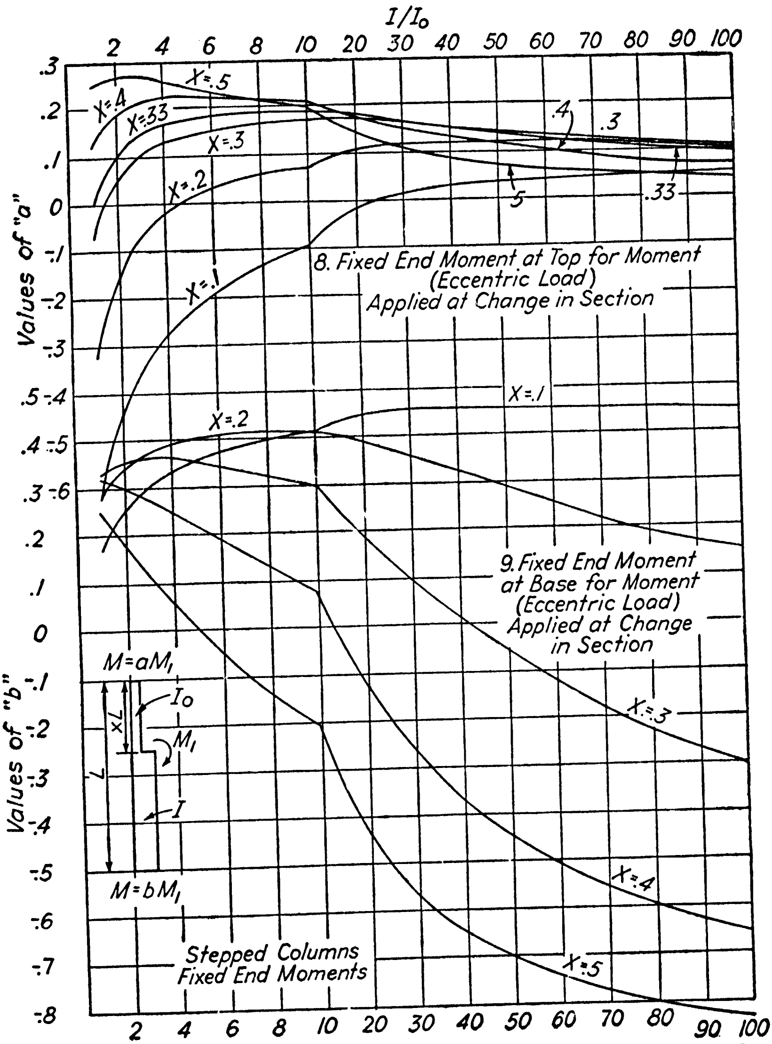
Courtesy of D. S. Ling.

Fixed-End Moments for a Uniform Load Acting on Beams and Columns with Sudden Change in Cross Section



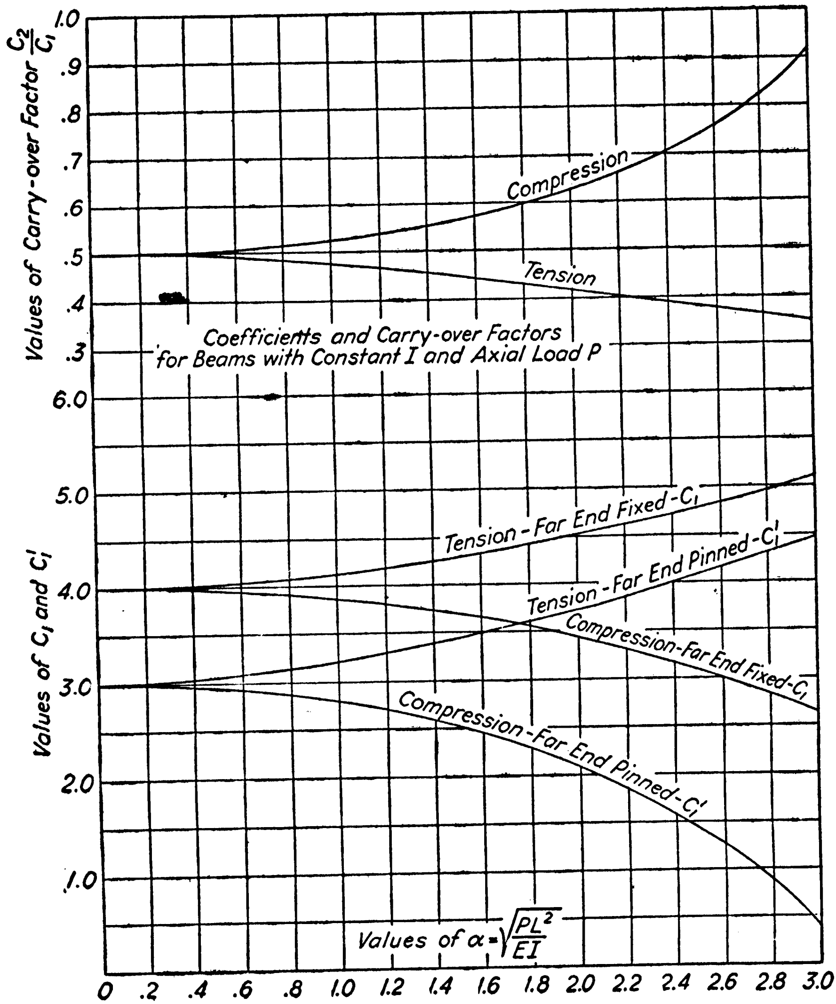
Courtesy of D. S. Ling.

FIXED-END MOMENTS FOR BEAMS AND COLUMNS WITH AN EXTERNAL MOMENT APPLIED AT CHANGE IN CROSS SECTION

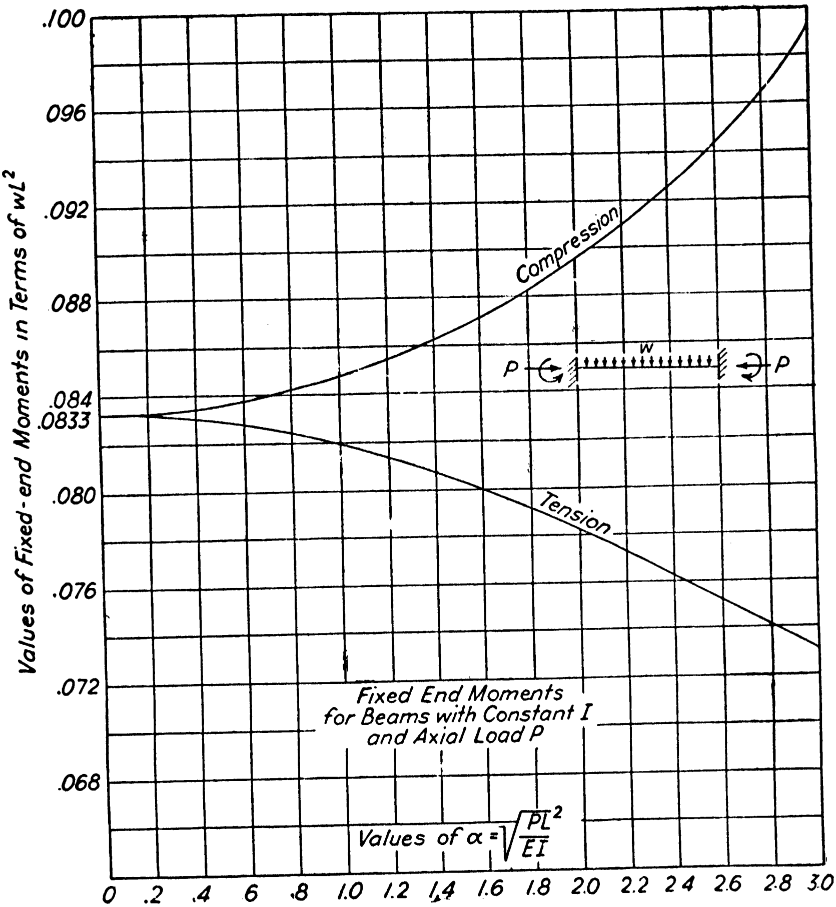


Courtesy of D. S. Ling

COEFFICIENTS FOR MEMBERS WITH AXIAL LOAD AND CONSTANT CROSS SECTION



FIXED-END MOMENTS FOR A UNIFORM LOAD APPLIED TO MEMBERS WITH AXIAL LOAD AND CONSTANT CROSS SECTION



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